M343/M

Third Level Course Examination 1995
Applications of Probability

Monday, 23 October, 1995 2.30pm–5.30pm

Time allowed: 3 hours

This examination is in **TWO** parts. Part I carries 40% of the total available marks and Part II carries 60%.

You should attempt **FOUR** questions from Part I: each question carries 10 marks. You should attempt **THREE** questions from Part II: the questions in this part carry 20 marks each.

Write your answers to Parts I and II in **separate** answer books. Please start each question on a new page, and cross out rough working.

At the end of the examination

Fasten together your answer books for Parts I and II, using the paper fastener provided. Check that you have written your name, personal identifier and examination number on **each** answer book used. **Failure to do so will mean that your work cannot be identified.**

PART I (Questions 1 to 6)

You should attempt **FOUR** *questions from this part of the examination, which carries 40% of the total available marks. Each question carries 10 marks.*

Question 1

During the night shift in a casualty department (10 pm to 6 am) patients may arrive requiring emergency treatment. The arrival of emergencies may be modelled as a non-homogeneous Poisson process with hourly rate

$$\lambda(t) = \frac{2t}{1+t^2}, \quad 0 \leq t \leq 8,$$

where t is the elapsed time in hours since 10 pm.

(i) Find the expected number of emergency admissions by t hours after 10 pm.

(ii) Find the expected number of emergencies

(a) during a night shift, (b) between midnight and 2 am.

(iii) Find the probability that not more than one patient arrives requiring emergency treatment during a night shift.

(iv) Find the probability that at least one emergency is admitted between midnight and 2 am.

Question 2

The locations of diseased trees in a wood are recorded and the numbers in each of 21 contiguous quadrats are counted. The data are as follows.

12	8	3	5	8	6	10
4	2	5	7	1	9	3
10	11	11	4	2	10	7

Commenting on the suitability of whatever test you adopt, use these data to investigate whether the diseased trees could reasonably be supposed to be randomly distributed in the wood. If your test suggests that this is not the case, say what sort of pattern you think they might follow.

Question 3

A six-state Markov chain has the following transition matrix.

$$\mathbf{P} = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \end{pmatrix}$$

(i) If the Markov chain is in state 2 at time 0, find the probability that it is again in state 2 at time 2.

(ii) If the Markov chain is in state 2 at time 0, find the probability that it is in state 6 at time 1 and in state 1 at time 2.

(iii) Classify the states of the Markov chain, identifying communicating classes, and stating whether each class is closed or not closed, recurrent or transient, and periodic or aperiodic; if periodic, state its period.

Question 4

Customers arrive at a cash dispenser at random at an average rate of three every ten minutes. If the machine is being used, then an arriving customer joins the queue.

(i) What condition must the mean service time satisfy for the equilibrium queue size distribution to exist?

(ii) If the service time has a uniform distribution, taking anything between 2 and 3 minutes, characterize the queue according to the standard notation. Find the mean queue size in this case.

(iii) Find the mean queue size if the service time is exponentially distributed with mean $2\frac{1}{2}$ minutes. In this case, for how long should a customer wishing to use the cash machine expect to have to queue (including the time spent using the machine)?

Question 5

Members of a stationary population live for a time which has probability density function

$$f(x) = kx^{k-1}, \quad 0 \leq x \leq 1,$$

where k is a positive constant and x is measured in years.

(i) Find the life table function $Q(x)$ for members of the population.

(ii) Find the mean lifetime of members of the population.

(iii) Find the mean age of members of the population.

(iv) What proportion of members of the population live for less than six months?

Question 6

An animal leaves its burrow and moves up and down a north-south hedgerow at the edge of a field foraging for food. Its distance north of its burrow t minutes after leaving the burrow is denoted by $X(t)$, and may be reasonably modelled as an ordinary Brownian motion $\{X(t); t \geq 0\}$ with diffusion coefficient $\sigma^2 = 2\,(\text{metres})^2$ per minute.

(i) Find the probability that after 2 minutes the animal is more than 3 metres north of its burrow.

(ii) If the animal is observed to be 2 metres north of its burrow after 5 minutes, find the probability that it is less than 3 metres north of its burrow after 10 minutes.

(iii) If the animal is observed to be 10 metres south of its burrow after half an hour, find the probability that it was north of its burrow after 20 minutes.

PART II (Questions 7 to 12)

You should attempt **THREE** *questions from this part of the examination, which carries 60% of the total available marks. Each question carries 20 marks: the mark allocation for each part of each question is shown beside each part thus:* [4].

Question 7

(a) The offspring random variable X in a population developing as a Galton–Watson branching process has probability distribution $P(X = x) = p_x$, $x = 0, 1, 2, \ldots$, where $p_0 > 0$, and mean μ. Describe briefly what might eventually happen to the evolving population in each of the following cases:

(i) $\mu < 1$,

(ii) $\mu = 1$,

(iii) $\mu > 1$. [3]

(b) In a Galton–Watson branching process starting with a single individual in generation zero, the offspring random variable X has a Poisson distribution with parameter 1.5.

(i) Find the mean and variance of the number of individuals in the second generation. [3]

(ii) Find the probability that the process becomes extinct

(a) by the second generation,

(b) at the third generation. [6]

(iii) Using your answer to part (a), say what might happen to the evolving population as time passes. Attach probabilities to whatever possibilities you have identified. [6]

(iv) If there were two individuals in generation zero, what would be the probability that the population eventually becomes extinct? [2]

Question 8

(a) Describe an example of a situation which could reasonably be modelled by a particle executing a simple random walk with two reflecting barriers. [2]

(b) A particle executes an unrestricted random walk starting at the origin; its position after n steps is denoted X_n. The ith step, Z_i, has the following distribution:

$$P(Z_i = 2) = p, \quad P(Z_i = -3) = q = 1 - p.$$

(i) Derive the probability distribution of X_n; that is, find the probability $P(X_n = x)$, and state the range of X_n. Explain all the steps in your derivation. [5]

(ii) Find the following in terms of p and q.

(a) $E(X_n)$ (b) $V(X_n)$ (c) $P(X_7 = 3)$ (d) $P(X_8 = 1)$ [7]

(iii) Calculate in terms of p and q the probabilities u_0, u_5 and u_{10} and hence find the probability that the particle returns to the origin for the first time after 10 steps. [6]

Question 9

(a) (i) Describe briefly one example of a random phenomenon that could reasonably be modelled as a simple birth process.

(ii) Describe briefly one example of a random phenomenon that could reasonably be modelled as an immigration-death process. [4]

(b) A partial differential equation for the probability generating function $\Pi(s,t)$ of the integer-valued random variable $X(t)$, which denotes the number of individuals alive at time t in a simple birth process $\{X(t); t \geq 0\}$ with birth rate β, is

$$\frac{\partial \Pi}{\partial t} = -\beta s(1-s)\frac{\partial \Pi}{\partial s}.$$

(i) Show that this partial differential equation has the general solution

$$\Pi(s,t) = \psi\left(\frac{s}{1-s}e^{-\beta t}\right).$$ [6]

(ii) Suppose that the number of individuals alive at time 0, when observation of the process commences, has a negative binomial distribution starting at $r = 2$ and with parameter p. Find the particular solution corresponding to this initial condition. [6]

(iii) Identify the distribution of $X(t)$ in this case and hence find the expected size of the population at time t. [2]

(iv) Suppose that at some time when there are k individuals in the population, they all suddenly cease giving birth and become liable to die; the lifetime of each of these individuals is exponential with mean $1/\nu$. Find the probability that the population dies out within a time w. [2]

Question 10

(a) Describe the main stages in the development of a disease after a susceptible individual is infected. What assumptions about these stages are made in

(i) the simple epidemic model, (ii) the general epidemic model? [5]

(b) In a family of three people (two children and their father), the father returns from work with a cold. Thereafter, the spread of the cold through the family may be modelled as a stochastic general epidemic. The infectious contact rate between members of the family is $\beta = 0.5$ per day, and any person who catches the cold remains infectious for a time which is exponentially distributed with mean 4 days.

(i) Find the probability distribution of the number of members of the family who never catch the cold. [6]

In fact, the whole family of three ends up catching the cold. The two children return to school while still infectious. There are 29 other children in their class, all of whom are susceptible to the cold. In this environment, the infectious contact rate is $\beta = 0.75$ per day.

(ii) Show that the epidemic parameter is equal to 10. [1]

(iii) Using a deterministic model, find the number of children who have the cold when the epidemic is at its height. How many children in the class escape without catching the cold? Draw a rough sketch showing the number of susceptibles and the number of infectives over the course of the epidemic. [8]

Question 11

A certain skin condition may be thought of as being due to a single recessive allele **a**. The 'normal' allele **A** is dominant. A population where this skin condition occurs is in Hardy–Weinberg equilibrium, the alleles **A** and **a** occurring in the proportions $\frac{3}{4} : \frac{1}{4}$. Mating is at random.

(i) What proportion of individuals have the skin condition? What proportion are carriers (genotype **Aa**)? [2]

(ii) Given that an individual does not have the skin condition, find the probability that he or she is

(a) a carrier; (b) normal (genotype **AA**). [4]

Adrian and Barbara, neither of whom have the skin condition, have two children, Carl and Deborah.

(iii) Find the probability that Carl

(a) is normal; (b) is a carrier; (c) has the skin condition. [7]

(iv) Find the probability that both children are carriers. [3]

(v) If Carl is known to have the skin condition, find the probability that Deborah does not have it. [4]

Question 12

The lifetime T (in years) of one of the components of an office photocopier may be described in terms of its hazard function

$$h(t) = \frac{6}{1+2t}, \quad t \geq 0.$$

(i) Find the survivor function for the components. What proportion of components last for less than three months? [4]

(ii) Say, giving reasons, whether these components are NBU, NWU or neither of these. [3]

(iii) Calculate the mean lifetime and the median lifetime of these components. [5]

Whenever a component fails, it is immediately replaced by a new one. The technician responsible for maintenance of all the office equipment initiates new monitoring procedures at a time when, as it happens, the photocopier is operating perfectly satisfactorily.

(iv) Obtain the probability density function of the waiting time until the next breakdown of the particular component in use when the monitoring procedures are initiated. Find the expected time until this component fails. [4]

(v) If the photocopier is used for eighteen months after the monitoring procedures are initiated, what is the expected number of new components that will be required in this time? [2]

(vi) Is the mean total lifetime of the component in the photocopier when the monitoring procedures are introduced longer or shorter than the mean lifetime of a new component chosen at random and installed in the machine? Give reasons for your answer. (*No calculations are required for this part of the question.*) [2]

[END OF QUESTION PAPER]