M343/U

The Open University

Third Level Course Examination 1994

Applications of Probability

Thursday, 27 October, 1994 2.30 pm – 5.30 pm

Time allowed: 3 hours

This examination is in **TWO** parts. Part I carries 40% of the total available marks and Part II carries 60%.

You should attempt **FOUR** questions from Part I: each question carries 10 marks. You should attempt **THREE** questions from Part II: the questions in this part carry 20 marks each.

Write your answers to Parts I and II in **separate** answer books. Please start each question on a new page, and cross out rough working.

At the end of the examination

Fasten together your answer books for Parts I and II, using the clip provided. Check that you have written your name, personal identifier and examination number on each answer book used. **Failure to do so will mean that your work cannot be identified.**

PART I (Questions 1 to 6)

You should attempt **FOUR** *questions from this part of the examination, which carries 40% of the total available marks. Each question carries 10 marks.*

Question 1

The positions of a particular species of plant in a meadow may be assumed to be reasonably modelled by a two-dimensional Poisson process with plant density $\lambda = 0.8$ per square metre.

(i) A plant is selected at random. What is the expected distance to the plant nearest to it? Find the probability that the nearest plant is more than 50 cm away.

(ii) A rectangular region 2 metres long by 1.5 metres wide is cordoned off. Find the probability that there are no plants in this region. What is the probability that there are more than three plants in the region?

(iii) The number of flowers on each plant has a $B(4, 0.6)$ distribution. How many flowers would you expect to find in the 2 metres by 1.5 metres region?

Question 2

Each morning, the assistant manager of a department store looks after the music department while the sales assistant takes his coffee break. The number of customers requiring attention during the coffee break has a Poisson distribution with parameter 4. The number of items purchased by each customer has a geometric distribution (starting at 0) with mean 2.

(i) Obtain the probability generating function of the number of items purchased by a customer. What proportion of customers make more than one purchase?

(ii) Find the probability generating function of the total number of items purchased during the coffee break. Hence find the mean number of items purchased during the coffee break.

(iii) Find the probability that no purchase is made during the coffee break.

Question 3

A particle executes an unrestricted simple random walk on the line with $p = q = \frac{1}{2}$. Its position at time n is represented by the random variable $X_n, n = 0, 1, 2, \ldots$. Its initial position is given by $X_0 = 0$.

(i) Find the values of the probabilities

(a) $P(X_5 = 0)$; (b) $P(X_6 = 0)$; (c) $P(X_5 = -1)$.

(ii) Find the probability that the particle returns to its starting point for the first time after 10 steps.

(iii) Find an approximate value for the probability that the particle is less than 10 units from its starting point after 100 steps.

Question 4

A particular type of colour blindness occurring within a population can be thought of as due to a single recessive allele **c** which is carried on the X chromosome; the normal allele **C** is dominant. The **C** and **c** allele proportions may be assumed to be the same in each sex with

$$P(\mathbf{C}) = p, \quad P(\mathbf{c}) = q = 1 - p.$$

Assume that the population reproduces generation by generation by a process of random mating.

(i) What proportion of men are colour blind? What proportion of women are carriers?

(ii) Angela, who is not colour blind but may be a carrier, marries Brian, who is not colour blind.

(a) What is the probability that Angela is a carrier?

(b) If Angela and Brian were to have a son, what is the probability that he would be colour blind?

(c) Angela and Brian have a son Desmond who is not colour blind. Given this information, find the probability that Angela is a carrier.

Question 5

The lifetime T (in years) of one of the components in a central heating system may be described in terms of its survivor function

$$Q(t) = 1 - \frac{t}{6}, \quad 0 \le t \le 6.$$

(i) Identify the distribution of T and hence write down the mean lifetime of these components.

(ii) Say, giving reasons, whether the component is NBU, NWU or neither of these.

(iii) After many years operation the heating system is replaced and the component is removed and put to use elsewhere. Write down the p.d.f. of the remaining lifetime of the component and hence calculate the probability that it will need replacing within 2 years.

Question 6

The AR(2) process $\{X_t;\ t = 0, 1, 2, \ldots\}$ is defined by

$$X_t = 0.3X_{t-1} + \alpha X_{t-2} + Z_t,$$

where $\alpha \neq 0$, $\{Z_t\}$ is a purely random process with mean 0 and variance 1, and each Z_t is independent of the preceding values of X.

(i) Between what bounds must α be for the process $\{X_t\}$ to be weakly stationary?

(ii) Assuming that all conditions for weak stationarity are satisfied and that $\alpha = 0.28$,

(a) find the autocorrelations ρ_0 and ρ_1;

(b) find the autocorrelation function ρ_τ.

PART II (Questions 7 to 12)

You should attempt **THREE** *questions from this part of the examination, which carries 60% of the total available marks. Each question carries 20 marks: the mark allocation for each part of each question is shown beside each part thus:* [4].

Question 7

(a) Give one example (different from that in part (b)) of a random phenomenon that could reasonably be modelled as a non-homogeneous Poisson process. Draw a rough sketch of the rate parameter $\lambda(t)$ for your example. [3]

(b) Over the course of an eight-hour day (9 am to 5 pm), customers visit a carpet shop according to a non-homogeneous Poisson process at an hourly rate

$$\lambda(t) = \tfrac{3}{8}t(8-t), \quad 0 \leq t \leq 8,$$

where t is the elapsed time in hours since 9 am.

(i) Find the mean number of customers to visit the shop per day. [3]

(ii) Find the probability that the first customer one day arrives after 10 am. [3]

(iii) One day, due to staff shortage, the shop is closed for lunch between 1 pm and 2 pm. Find the probability that exactly two potential customers arrive to find the shop closed. [4]

(iv) Another day, the shop has exactly three customers between 1 pm and 3 pm. Find the probability that two of them arrive before 2 pm and the third after 2 pm. [7]

Question 8

Once a week, Alan visits his local cinema; he always buys an ice cream or sorbet to eat while watching the film. The cinema sells vanilla tubs (V), chocolate nut sundaes (C) and lemon sorbet (L). What Alan buys on any visit depends on what he had the previous week: the pattern of his weekly purchases may be modelled by a Markov chain with the following transition matrix.

$$\mathbf{P} = \begin{matrix} V \\ C \\ L \end{matrix} \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

(i) If he buys a chocolate nut sundae this week, find the probability that he buys (a) a lemon sorbet next week and a vanilla tub the week after that, (b) a vanilla tub in two weeks' time. [4]

(ii) On his first visit after a holiday away, he is equally likely to buy any of the three. Find the probability that he buys a lemon sorbet (a) on his second visit after a holiday away; (b) on his third visit after a holiday. [4]

(iii) In the long run, on what proportion of his visits to the cinema does he buy (a) a vanilla tub, (b) a chocolate nut sundae, (c) a lemon sorbet? [6]

(iv) One week Alan is observed to buy a lemon sorbet. What is the expected number of weeks until he next buys a lemon sorbet? [2]

Suppose that all customers who buy ice creams or sorbets alter their choices from one visit to the next in the same way as Alan does.

(v) How would you interpret an initial distribution of $\mathbf{a}^{(0)} = (\frac{1}{3}\ \frac{1}{3}\ \frac{1}{3})$? [2]

(vi) How might the cinema manager interpret the limiting distribution? [2]

Question 9

(a) An immigration-birth process with arrival rate λ and birth rate β may be described by the probability statement

$$P(X(t+\delta t) = x+1 | X(t) = x) = (\lambda + \beta x)\delta t + o(\delta t),$$

where $X(t)$ denotes the size of the population at time t.

Suppose that at time 0 a population growing according to this rule is of size 1. If $\lambda = 2\beta$, find in terms of β the mean and variance of the waiting time until a population size of 5 is attained. [4]

(b) A partial differential equation for the probability generating function $\Pi(s,t)$ of the integer-valued random variable $X(t)$, which denotes the number of individuals alive at time t in an immigration-birth process $\{X(t); t \geq 0\}$, is

$$\frac{\partial \Pi}{\partial t} = -\lambda(1-s)\Pi - \beta s(1-s)\frac{\partial \Pi}{\partial s}.$$

(i) Rewrite this equation in Lagrange form and identify the functions f, g and h. [2]

(ii) Write down the auxiliary equations and show that their two solutions may be written

$$c_1 = \frac{s}{1-s}e^{-\beta t}, \quad c_2 = s^{\lambda/\beta}\Pi.$$ [7]

(iii) Write down the general solution for the partial differential equation for $\Pi(s,t)$. [1]

(iv) Show that the particular solution for $\Pi(s,t)$, given that there are no individuals alive at time $t = 0$, is given by

$$\Pi(s,t) = \left(\frac{e^{-\beta t}}{1-(1-e^{-\beta t})s}\right)^{\lambda/\beta}.$$ [4]

(v) If $\lambda = 2\beta$, find the probability that there are two individuals alive at time t, given that there are no individuals alive at time $t = 0$. [2]

Question 10

Two assistants are on duty in a post office. Customers may be assumed to arrive independently and at random at an average rate of 30 per hour. If an assistant is free, then an arriving customer is served immediately; otherwise a central queue is formed. The service time for each assistant may be assumed to be exponentially distributed with mean 3 minutes.

(i) Write down the specification of this queue, and show that the traffic intensity ρ is $\frac{3}{4}$. [2]

Assume that the queue is in equilibrium.

(ii) Find the proportion of the time that both assistants are idle. [6]

(iii) What proportion of customers have to wait to be served? [3]

(iv) Find the proportion of the time that there are more than two people waiting to be served. [3]

(v) Show that the probability generating function of the equilibrium queue size is given by

$$\Pi(s) = \frac{1}{14}\left(2 + 3s + \frac{9s^2}{4-3s}\right)$$

and hence evaluate the mean equilibrium queue size. [6]

Question 11

(a) Briefly describe the main similarities and differences between the threshold phenomena for the stochastic general epidemic model and the deterministic general epidemic model. [4]

(b) In a hostel housing 25 people, one person catches an infectious disease. The spread of the disease through the hostel can be modelled by a stochastic general epidemic. The infectious contact rate is $\beta = 3$ per day. Anyone who catches the disease is infectious for a time which is exponentially distributed with mean 2 days.

(i) Show that the epidemic parameter ρ is equal to 4. [1]

(ii) Calculate the probability that fewer than two of the initially uninfected residents of the hostel catch the disease before the epidemic dies out. [6]

(iii) What approximately is the probability that there is only a minor outbreak of the disease? [2]

(iv) Using a deterministic model, find

(a) the maximum number of residents ill at any one time;

(b) the number of residents who do not catch the disease.

What do your answers tell you about the spread of the disease through the hostel? [7]

Question 12

The age-specific death rate for the members of a particular stationary bird population is given by

$$h(x) = \frac{3}{10-x}, \quad 0 \le x < 10,$$

where x, the age of a bird, is measured in years.

(i) Obtain the life table function $Q(x)$ for members of the population. [3]

(ii) Find the median age at which birds die. [2]

(iii) Find the mean lifetime of birds in this population. What proportion of birds live more than twice the mean lifetime? [3]

(iv) Find the median age of the birds in the population. [4]

(v) What proportion of birds live for less than two years? At any time, what proportion of the birds in the population are less than two years old? [3]

(vi) Draw a rough sketch of the p.d.f. of the age-distribution of the population. Now suppose that the population with the life table function of part (i) is stable but not stationary. Describe briefly how the age-distribution for the population would differ from that of the stationary population if the population is (a) growing, (b) declining. Illustrate your answer in each case by drawing a rough sketch of the p.d.f. of the age-distribution. (*No calculations are required for this part of the question*). [5]

[END OF QUESTION PAPER]