M343/S

Third Level Course Examination 1993

Applications of Probability

Wednesday, 27 October, 1993 — 2.30 pm – 5.30 pm

Time allowed: 3 hours

This examination is in **TWO** parts. Part I carries 40% of the total available marks and Part II carries 60%.

You should attempt **FOUR** questions from Part I: each question carries 10 marks. You should attempt **THREE** questions from Part II: the questions in this part carry 20 marks each.

Write your answers to Parts I and II in **separate** answer books. Please start each question on a new page, and cross out rough working.

At the end of the examination

Fasten together your answer books for Parts I and II, using the clip provided. Check that you have written your name, personal identifier and examination number on each answer book used. **Failure to do so will mean that your work cannot be identified.**

PART I (Questions 1 to 6)

You should attempt **FOUR** *questions from this part of the examination, which carries 40% of the total available marks. Each question carries 10 marks.*

Question 1

Letters for translation are received by an official according to a Poisson process with mean interarrival time two hours. He occasionally makes mistakes during translation: Y, the number of mistakes per letter, has a geometric distribution, $G_0(\frac{1}{3})$.

(i) Find the mean and variance of $S(t)$, the total number of mistakes the official makes in t hours.

(ii) Show that $\Pi(s)$, the probability generating function of $S(t)$, is given by

$$\Pi(s) = \exp\left(-\tfrac{1}{2}t(1-s)/(3-s)\right).$$

(iii) Hence find the probability that the official makes at least one mistake between 9am and 1pm.

Question 2

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In an investigation into the disposition of a species of wild flower in a meadow, a sample of point-to-nearest-object distances (R-distances) is obtained, and a sample of the same number of object-to-nearest-object distances (S-distances). The following data (distances in metres) are collected.

R-distances	1.6	2.1	1.8	1.4	0.9	1.5
S-distances	2.9	3.2	3.1	2.3	2.6	3.3

Use this information to investigate whether the flowers may be regarded as being randomly located over the meadow. If your investigation suggests that they are not randomly located, then say how you think the flowers are located.

Question 3

Every Friday, Edward borrows a book from his local library to read over the weekend; he always reads either biography (B), classical fiction (C) or detective fiction (D). The type of book that he borrows depends on the type of book that he read the previous weekend: the pattern of his book borrowing may be modelled by a Markov chain with the following transition matrix.

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$$\mathbf{P} = \begin{array}{c} B \\ C \\ D \end{array} \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

(i) If he reads a biography one weekend, find the probability that he reads detective fiction the next weekend and classical fiction the weekend after that.

(ii) According to this model, on what proportion of weekends in the long run does he read (a) classical fiction (b) detective fiction?

(iii) If he reads a biography one weekend, what is the expected number of weeks until he next reads a biography?

Question 4

A population is growing according to a simple birth process with birth rate β. At time 0, there is one individual in the population.

(i) Write down the distribution of T_n, the time between the $(n-1)$th and the nth births.

7 (ii) Find in terms of β the mean and variance of the waiting time until a population size of 6 is attained.

(iii) Write down the distribution of $X(t)$, the size of the population at time t. Hence find the c.d.f. of W_n, the waiting time until the nth birth occurs (that is, until the population reaches size $n+1$).

Question 5

All that is known about a particular stationary population is that the age-specific death rate for its members is given by the formula

$$h(x) = \beta^2 x, \quad x \geq 0,$$

where β is a positive constant.

11 (i) Find the life table function $Q(x)$ for members of this population. Identify the lifetime distribution of an individual, and hence write down the expectation of life at birth.

(ii) Find the median lifetime for members of the population.

(iii) Find the mean age of members of this population.

Question 6

The waiting time T between consecutive occurrences of an event E in a discrete-time renewal process has the probability distribution

$$P(T=1) = 0.5, \qquad P(T=2) = 0.5.$$

13 (i) Find the generating function $U(s)$ for this process, and hence or otherwise find the probabilities u_2 and u_3.

(ii) Denoting by W_5 the waiting time to the fifth event, find the probability $P(W_5 = 7)$.

(iii) If an observer arrives after the renewal process has been running for a long time, what is the approximate probability that an event occurs at the next time point?

PART II (Questions 7 to 12)

You should attempt **THREE** *questions from this part of the examination, which carries 60% of the total available marks. Each question carries 20 marks: the mark allocation for each part of each question is shown beside each part thus:* [4].

Question 7

(a) Let $\Pi(s)$ be the p.g.f. of the offspring distribution in a Galton-Watson branching process $\{Z_n; n = 0, 1, 2, \ldots\}$ starting with a single ancestor in generation zero. If $\Pi_n(s)$ denotes the p.g.f of Z_n, the population size at generation n, show that

$$\Pi_n(s) = \Pi(\Pi_{n-1}(s)).$$

State briefly any results you use. [4]

(b) In a Galton-Watson branching process starting with a single individual in generation zero, the offspring distribution is binomial, $B(3, \frac{1}{2})$.

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(i) Write down the mean and variance of the number of individuals in the first generation, and calculate the mean and variance of the number of individuals in the second generation. [4]

(ii) Find the probability that the process becomes extinct at the third generation. [5]

(iii) Calculate to 4 decimal places the probability that the process eventually becomes extinct. [5]

(iv) If there were four individuals in generation zero, what would be the probability that the process eventually becomes extinct? [2]

Question 8

(a) Give one example each of a situation which could reasonably be modelled by

(i) a particle executing a random walk on the line with two reflecting barriers,

(ii) a particle executing a random walk on the line with two absorbing barriers. [4]

(b) A particle executes a simple unrestricted random walk on the line, a step to the right of length 1 occurring with probability $p = 0.4$, and a step to the left of length 1 occurring with probability $q = 1 - p = 0.6$. Its position at time n is represented by the random variable $X_n, n = 0, 1, 2, \ldots$. Its initial position is given by $X_0 = 0$.

(i) Show that the distribution of X_n is given by

$$P(X_n = k) = \binom{n}{(n+k)/2} p^{(n+k)/2}\, q^{(n-k)/2}$$

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for $k \in \{-n, -n+2, \cdots, n-4, n-2, n\}$. [4]

(ii) Find the values of the following probabilities:

(a) $P(X_6 = 0)$ (b) $P(X_7 = 0)$ (c) $P(X_7 = -1)$ [3]

(iii) Find the probability that the particle returns to the origin for the first time after 6 steps. [2]

(iv) After the particle has been moving for some time, it is observed to be located at the point +4.

(a) Find the probability that it ever returns to the origin.

(b) If it returns to the origin, what is the probability that it returns to the point +4 at some later time? [4]

(v) Find the probability that, starting from the origin, the particle visits the point +6 before it visits the point −3. [3]

Question 9

A general birth and death process $\{X(t);\ t \geq 0\}$ is characterized by the following probability statements

$$\left.\begin{array}{l} P(X(t+\delta t) = x+1 | X(t) = x) = \beta_x \delta t + o(\delta t) \\ P(X(t+\delta t) = x-1 | X(t) = x) = \nu_x \delta t + o(\delta t) \end{array}\right\} x = 0, 1, 2, \ldots.$$

(i) Write down the Kolmogorov forward equations for the process and hence show that if an equilibrium distribution $\{p_x\}$ exists for the random process, then

$$p_1 = \frac{\beta_0}{\nu_1} p_0, \quad p_2 = \frac{\beta_0 \beta_1}{\nu_1 \nu_2} p_0.$$

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Write down an explicit general formula for p_x in terms of the parameters $\{\beta_x\}$ and $\{\nu_x\}$ and the probability p_0, and state clearly the condition for the equilibrium distribution to exist. [5]

The furniture department of a large store has three assistants on duty at a time. Customers requiring assistance arrive independently and at random at an average rate of one every ten minutes. If all three assistants are busy, then an arriving customer goes away. The time taken to serve a customer is an exponentially distributed random variable with mean 20 minutes. The number of customers in the department at time t is $X(t)$.

(ii) Write down the values of the parameters $\{\beta_x\}$ and $\{\nu_x\}$ for this process, for all $x = 0, 1, 2, \ldots$. [4]

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(iii) Find the equilibrium probability distribution of the number of customers in the department. [6]

(iv) (a) For what proportion of the time are all the assistants idle?

(b) What proportion of arriving customers go away because no assistant is free to serve them?

(c) What is the mean number of idle assistants at any one time? [5]

Question 10

(a) Identify the main differences between the simple epidemic model and the general epidemic model. [4]

(b) In a family of total size 6, initially two people are suffering from a disease which is incurable, but not serious; the other four are susceptible. Infectives are not isolated. The infective contact rate is $\beta = 0.5$ per day.

(i) Suggest a stochastic model which may be used to represent the spread of the disease in the family. [1]

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(ii) Find the expected waiting time until the whole family has the disease. [5]

(iii) Find the standard deviation of the waiting time in part (ii). [3]

(c) In another family of total size 6, initially two people are suffering from a different disease; the other four members of the family are susceptible. An individual who catches this disease remains infectious for a time which is exponentially distributed with mean 5 days. The infective contact rate is $\beta = 0.5$ per day.

(i) Show that the epidemic parameter ρ is equal to 2. [2]

(ii) Use a stochastic model to calculate the probability that one and only one of the four susceptible members of the family catches the disease before the epidemic dies out. [5]

Question 11

In a large population of animals reproducing in discrete generations by a process of random mating, eye colour is controlled by a single gene. Doubly dominant animals **AA** and heterozygotes **Aa** have green eyes; homozygous recessives **aa** have yellow eyes. In the population as a whole, 96% of the animals have green eyes.

(i) What proportion of the animals are 'pure' green-eyed (doubly dominant **AA**)? [3]

(ii) What proportion of green-eyed animals are 'pure' green-eyed? [3]

In an attempt to breed only green-eyed animals, a large number of green-eyed animals are removed to a new habitat. (All the yellow-eyed animals are left behind.)

(iii) Assuming that mating habits in the new habitat remain unaltered, explain what actually happens, mentioning any theorem you use. [7]

(iv) After several generations of random mating in the new habitat, two green-eyed animals are selected at random.

(a) Calculate the probability that both these animals are 'pure' green-eyed (**AA**).

(b) These two green-eyed animals are crossed to produce a single offspring. Calculate the probability that this offspring is 'pure' green-eyed (**AA**).

(c) Given that the offspring in (b) is 'pure' green-eyed, calculate the probability that the parents are both 'pure' green-eyed. [7]

Question 12

(a) Give one example each of (i) a phenomenon which might be modelled by Brownian motion with drift, and (ii) a diffusion where a Brownian bridge would be an appropriate model. [4]

(b) The price of a share in a company fluctuates in such a way that the random variable

$$Y(t) = \frac{\text{price at time } t}{\text{price at time } 0}$$

may be modelled as a geometric Brownian motion, derived from the ordinary Brownian motion $\{X(t);\ t \geq 0\}$ with diffusion coefficient $\sigma^2 = \frac{1}{4}$ per year through the relationship $Y(t) = \exp X(t)$.

(i) Find the probability that after three months the price of the share will be more than 20% up on its price at time 0. [5]

(ii) The price after nine months is observed to be the same as the price at time 0. Find the probability that after six months the price was more than 30% up on its price at time 0. [6]

(iii) Given the price of the share at time 0, find an expression for the probability $P(W_2 \leq w)$, where W_2 is the waiting time in years until the share price doubles. Hence find the probability that the share price does not double in the first six months. [5]

[END OF QUESTION PAPER]