Question 5

Use Abel's identity with $a(n) = n \log n$ when n is prime, and 0 otherwise, and with '$f(x)$' replaced by $1/\log x$. 1

Then

$$\sum_{p \leq x} (f(p) \log p.(1/\log p))$$

$$= \frac{1}{\log x} \sum_{p \leq x} f(p) \log p + \int_2^x (\sum_{p \leq t} f(p) \log p) \frac{dt}{t \log^2 t}.$$ 2

So $$\sum_{p \leq x} f(p) = x - \frac{x}{\log x} + O\left(\frac{1}{\log x}\right) + \int_2^x \left(\frac{1}{\log t} - \frac{1}{\log^2 t}\right) dt + O\left(\int_2^x \frac{dt}{t \log^2 t}\right).$$ 1

But $$\int_2^x \frac{dt}{\log t} = \left[\frac{t}{\log t}\right]_2^x + \int_2^x \frac{dt}{\log^2 t} \quad \text{(integrating by parts)}$$

so $$\int_2^x \left(\frac{1}{\log t} - \frac{1}{\log^2 t}\right) dt = \frac{x}{\log x} - \frac{2}{\log 2}.$$ 2

Also $$\int_2^x \frac{dt}{t \log^2 t} = \left[-\frac{1}{\log t}\right]_2^x = K - \frac{1}{\log x}.$$ 2

Thus $$\sum_{p \leq x} f(p) = \left(x - \frac{x}{\log x}\right) + \left(\frac{x}{\log x} - \frac{2}{\log 2}\right) + O(1)$$

$$= x + O(1).$$ 2

[Total: 10 marks]

Question 6

(a) The remainder is 1, by the Euler-Fermat theorem $a^{\phi(m)} \equiv 1 \pmod{m}$ with $a = 5$ and $m = 16$. 2

(b) $98^{76} \equiv 7^{76} \equiv 7^4 = (49)^2 \equiv (-3)^2 \equiv 9 \pmod{13}$. 4 (for any method)

(c) If $3.(16!) \equiv k \pmod{19}$, then $18! \equiv 6.17k \pmod{19} \equiv 7k \pmod{19}$. 2

But, by Wilson's theorem, 1

$$18! \equiv -1 \pmod{19},$$

so $7k \equiv -1 \pmod{19}$, and $k = 8$. 1

[Total: 10 marks]