Question 7

(i) Add 5: $2 \longrightarrow 7 \longrightarrow 12 \longrightarrow 17 \longrightarrow 22 \longrightarrow \mathbf{27}$ $(\equiv 3 \bmod 6)$ 1

Add 30: $27 \longrightarrow 57 \longrightarrow 87 \longrightarrow \mathbf{117}$ $(\equiv 5 \bmod 7)$. 1

So $x \equiv 117 \pmod{210}$. 1

OR With the notation of Apostol, page 118, $M = 210, M_1 = 42, M_2 = 35,$ $M_3 = 30, M_1' = 3, M_2' = 5, M_3' = 4.$ 1

So $x = 2.42.3 + 3.55.5 + 5.30.4 = 1377 \equiv 117 \pmod{210}$. 2

(ii) The congruence $x^2 - 12x - 8 \equiv 0 \pmod 7$ has solutions $r = 2$ and $r = 3$. 1

If $r = 2$, let $a = 2 + 7q$, in the notation of Apostol, page 122.
Here $f(r) = -28$, $k = -4$, $f'(r) = -8$, $-8q - 4 \equiv 0 \pmod 7$. 3
So $q = 3$, and $\mathbf{a = 23}$.

If $r = 3$, let $a = 3 + 7q$.
Here $f(r) = -35$, $k = -5$, $f'(r) = -6$, $-6q - 5 \equiv 0 \pmod 7$. 3
So $q = 5$, and $\mathbf{a = 38}$.
Thus, $x \equiv 23$ or $38 \pmod{49}$.

[*Total: 10 marks*]

Question 8

(i) We have the following table. (All other values are 0.)

n	1	5	7	11	13	17
χ_1	1	1	1	1	1	1
χ_2	1	-1	1	-1	1	-1
χ_3	1	ω	ω^2	ω^5	ω^4	-1
χ_4	1	ω^2	ω^4	ω^4	ω^2	1
χ_5	1	ω^4	ω^2	ω^2	ω^4	1
χ_6	1	ω^5	ω^4	ω	ω^2	-1

$\omega^6 = 1$
$\omega = \frac{1}{2}(1 + i\sqrt{3})$ 2 (for χ_1, χ_2)

6 (for χ_3 to χ_6)

(To construct this, notice that $\chi(5)\chi(5) = \chi(7)$, $\chi(5)\chi(17) = \chi(13)$, etc.)

(ii) $L(1, \chi_2) = (1 - \frac{1}{5}) + (\frac{1}{7} - \frac{1}{11}) + (\frac{1}{13} - \frac{1}{17}) + (\frac{1}{19} - \frac{1}{23}) + \cdots.$

Since each bracket is positive, $L(1, \chi_2) > 0$. 2

[*Total: 10 marks*]

Question 9

(i) $(823|999) = (823|3)^3(823|37)$ 1

$= (823|3)(9|37)$ 1

$= (1|3)(3|37)^2 = 1.$ 1

(ii) If $p \equiv 1 \pmod 4$, then $(11|p) = (p|11) = 1$ if $p \equiv 1, 4, 9, 5$ or $3 \pmod{11}$. 2
So $p \equiv \mathbf{1}, \mathbf{37}, \mathbf{9}, \mathbf{5}$ or $\mathbf{25} \pmod{44}$. 2

If $p \equiv 3 \pmod 4$, then $(11|p) = (-p|11) = 1$ if $p \equiv 2, 6, 7, 8$ or $10 \pmod{11}$. 1
So $p \equiv \mathbf{35}, \mathbf{39}, \mathbf{7}, \mathbf{19}$ or $\mathbf{43} \pmod{44}$ 2

[*Total: 10 marks*]