Question 6

Let H and K be groups, and let the group G be the direct product $H \times K$.

Consider each statement below, and state whether it is true or false, giving a brief proof or counter-example, as appropriate.

(i) If H and K are cyclic, then G is cyclic. [3]

(ii) If H and K are abelian, then G is abelian. [3]

(iii) If H and K are simple, then G is simple. [4]

Question 7

(i) For each of the following field extensions, state, with brief reasons, whether or not it is algebraic.

(a) $\mathbf{Q}(\cos \pi/4) : \mathbf{Q}$ [2]

(b) $\mathbf{Q}(\pi^2) : \mathbf{Q}$ [2]

(ii) For each of the following finite field extensions, state, with brief reasons, whether or not it is normal.

(a) $K : \mathbf{Z}_7$, where $K = \mathbf{Z}_7[t]/\langle t^2 + t + 4\rangle$. [2]

(b) $\mathbf{Q}(\sqrt[7]{2}, \sqrt{7}) : \mathbf{Q}$ [2]

(c) $\mathbf{Q}(\sqrt[7]{2}, \sqrt{7}, \omega) : \mathbf{Q}$, where $\omega^7 = 1, \omega \neq 1$. [2]

Question 8

Let K and L be fields with $K \subseteq L$. Let $\alpha \in L$ and let the minimum polynomial of α over K have degree n.

(i) Prove that $K(\alpha^2) \subseteq K(\alpha)$. [1]

(ii) Prove that α^2 is algebraic over K. [2]

(iii) Suppose that n is *odd*. Prove that $K(\alpha^2) = K(\alpha)$. [5]

(iv) Give an example with $K = \mathbf{Q}$ such that K is a proper subset of $K(\alpha^2)$ and $K(\alpha^2)$ is a proper subset of $K(\alpha)$. [2]

Question 9

Let K be the field $\mathbf{Q}(i, \sqrt{5})$, and let G be the Galois group $\Gamma(K : \mathbf{Q})$.

(i) Find G, specifying for each element of G its effect on i and on $\sqrt{5}$. [5]

(ii) Find the fixed subfield of each subgroup of G. [5]

Question 10

Suppose that $M : K$ is a finite field extension and L is a field such that $K \subseteq L \subseteq M$ and $L : K$ is a normal extension.

(i) Suppose $\sigma \in \Gamma(M : K)$. Prove that the restriction $\sigma|_L$ of σ to L is a K-automorphism of L. [2]

(ii) Prove that the function

$$\begin{aligned} \phi : \Gamma(M : K) &\longrightarrow \Gamma(L : K) \\ \sigma &\longmapsto \sigma|_L \end{aligned}$$

is a group homomorphism, and find its kernel. [3]

(iii) Hence show that the quotient $\Gamma(M : K)/\Gamma(M : L)$ is isomorphic to a subgroup of $\Gamma(L : K)$. [1]

(iv) By considering the extension $\mathbf{Q}(\sqrt[4]{2}) : \mathbf{Q}$, or otherwise, show that the quotient

$$\Gamma(M : K)/\Gamma(M : L)$$

may be isomorphic to a proper subgroup of $\Gamma(L : K)$. [4]