(iv) Dimension … 3, from above, if we need it.

Might be useful for Degree Theorem. Tempting to assume $\{1, \sqrt{2}, i, i\sqrt{2}\}$ is a basis for W, but not necessary.

$[K : \mathbf{Q}] \leq 4$ since $\{1, \sqrt{2}, i, i\sqrt{2}\}$ spans K over $\mathbf{Q}$. Since $[W : \mathbf{Q} = 3]$ and $W \subseteq K$, we must have $[K : \mathbf{Q}] \geq 3$. But $V \subseteq K$ so, by the Degree Theorem, 2 divides $[K : \mathbf{Q}]$. Hence $[K : \mathbf{Q}] = 4$. If W were a field we would have $\mathbf{Q} \subseteq W \subseteq K$ and $[K : \mathbf{Q}] = [K : W][W : \mathbf{Q}]$, i.e. $4 = [K : W] \times 3$. Since $[K : W]$ must be an integer, this is a contradiction. Hence W is not a field.

Question 4

(i) Permutations are conjugate if and only if they have the same disjoint cycle decomposition, so look for all of form $(..)(..)$.

Permutations conjugate to $(12)(34)$ are those with disjoint cycle form $(ab)(cd)$, i.e.

$$(12)(34), (13)(24), (14)(23).$$

(ii) Use $|x^G||C_G(x)| = |G|$.

Since $|\mathbf{S}_4| = 24$ and $|((12)(34))^{\mathbf{S}_4}| = 3$, we have

$$|C_{\mathbf{S}_4}((12)(34))| = 8.$$

(iii) First principles, or quote knowledge of subgroups of $\mathbf{S}_4$.

$\mathbf{S}_4$ has no cyclic subgroup of order 8 since $\mathbf{S}_4$ has no elements of order greater than 4. Hence $C_{\mathbf{S}_4}((12)(34))$ is not cyclic.

(iv) Try finding the order first. $\mathbf{S}_4$ has a number of subgroups of order 4.

There are $\dfrac{4 \times 3 \times 2 \times 2}{4} = 6$ 4-cycles in $\mathbf{S}_4$, so

$$|(1234)^{\mathbf{S}_4}| = 6$$

and hence

$$|C_{\mathbf{S}_4}(1234)| = 4.$$

But (1234) and all its powers commute with (1234), which gives a cyclic subgroup of order 4.

Since (1234) and its powers commute with (1234), we have

$$\langle(1234)\rangle \subseteq C_{\mathbf{S}_4}((1234)).$$

But $|\langle(1234)\rangle| = 4$, so

$$C_{\mathbf{S}_4}((1234)) = \langle(1234)\rangle.$$

Question 5

Direct products involve showing that the product of the subgroups is the whole group, and that the subgroups are normal with trivial intersection.
$H \cap N$ is normal in G so $G/(H \cap N)$ can be formed. The Correspondence Theorem (6.6.6) looks useful.

General strategy—turn each question about $G/(H \cap N)$ into a question about H and N in G; use each of the given pieces of information about H and N.

Normality Since H and N are normal subgroups of G, each containing $H \cap N$, Theorem 6.6.6 shows that $H/(H \cap N)$ and $N/(H \cap N)$ are normal in $G/(H \cap N)$.

Trivial intersection If

$$x(H \cap N) \in (H/(H \cap N)) \cap (N/(H \cap N)),$$

then $x \in H$ and $x \in N$, i.e.

$$x \in H \cap N.$$

Thus $x(H \cap N) = H \cap N$, the identity element of $G/(H \cap N)$.

Product all of $G/(H \cap N)$ Any $x(H \cap N)$ in $G/(H \cap N)$ has $x \in G$ so, since $G = HN$, we have $x = hn$ for some h in H, n in N. Thus

$$\begin{aligned} x(H \cap N) &= hn(H \cap N) \\ &= (h(H \cap N))(n(H \cap N)). \end{aligned}$$

So $G/(H \cap N) \subseteq (H/(H \cap N))(N/(H \cap N))$. The reverse inclusion is clear. Thus

$$G/(H \cap N) = (H/(H \cap N)) \times (N/(H \cap N)).$$