Question 14

(i) The given information is unusual in that n is prime as well as p. Degree of minimum polynomial is the same as degree of $\mathbf{Z}_p(\alpha) : \mathbf{Z}_p$. Subfields of K have the form $\mathbf{GF}(p^r)$ for r dividing n. That's it.

$\mathbf{Z}_p(\alpha)$ is a subfield of $K = \mathbf{GF}(p^n)$. Hence $\mathbf{Z}_p(\alpha) = \mathbf{GF}(p^r)$ with r a divisor of n, by Theorem 15.4.1. Since n is prime, $r = 1$ or $r = n$. Since the degree of the minimum polynomial is $[\mathbf{Z}_p(\alpha) : \mathbf{Z}_p]$, we have $\mathbf{Z}_p(\alpha) = \mathbf{Z}_p$ or K. Hence $[\mathbf{Z}_p(\alpha) : \mathbf{Z}_p] = 1$ or n. The result follows.

(ii) For degree 1, must be in bottom field.

By (i), anything outside bottom field gives degree 3.

$1 \in \mathbf{Z}_2$, so the minimum polynomial of 1 over $\mathbf{Z}_2$ is $t - 1$, of degree 1.

Let $\beta = t + \langle t^3 + t + 1 \rangle \in K$. Then β is not in $\mathbf{Z}_2$, so the minimum polynomial of β over $\mathbf{Z}_2$ has degree 3.

Question 15

Assemble what's known from the given information. $L = K(\alpha)$ for some α, not separable over K. We also know that K must have characteristic p for some prime p, and that α has some exponent e over K.

If we find that all zeros of the minimum polynomial of α over K are the same, then $\Gamma(L : K)$ must be trivial.

First, K is of characteristic p, for some prime p. We have $L = K(\alpha)$ for some $\alpha \in L \backslash K$. Since $L : K$ is purely inseparable, α is inseparable over K.

By Handbook, Section 16.2, Result 9, there is a non-negative integer e such that the minimum polynomial of α over K is

$$\begin{aligned} m(t) &= t^{p^e} - \alpha^{p^e} \\ &= (t - \alpha)^{p^e} \end{aligned}$$

since K has characteristic p. Hence m has all its zeros equal to α. Thus, since any element $\sigma \in \Gamma(L : K)$ maps α to a zero of m,

$$\sigma(\alpha) = \alpha.$$

But any such σ is entirely determined by its effect on α, so $\Gamma(L : K)$ is trivial.

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