Question 6

(i) Try $\mathsf{C}_2 \times \mathsf{C}_2$, $\mathsf{C}_2 \times \mathsf{C}_3$ ... for ideas.

This is not true.

For example, take $H = K = \mathsf{C}_2$.

Then $G = \mathsf{C}_2 \times \mathsf{C}_2 \cong \mathsf{V}$, which is not cyclic, because it contains no elements of order 4.

(ii) Seems plausible, so try to prove it.

This is true.

Typical elements of G are (h, k) and (h', k'), where $h, h' \in H$ and $k, k' \in K$.

$$\begin{aligned}(h,k)(h',k') &= (hh', kk') \\ &= (h'h, k'k) \quad \text{because } H \text{ and } K \text{ are abelian} \\ &= (h',k')(h,k).\end{aligned}$$

Thus G is abelian.

(iii) Aha! Did this in *Unit 12*. Direct products always have normal subgroups isomorphic to those you first thought of, so unless either of those is trivial we're home. Take an easy example.

This is not true.

For example, take $H = K = \mathsf{C}_2$. Then H and K are both simple, but $G\,(\cong \mathsf{V})$ is not simple because it has normal subgroups of order 2.

Question 7

(i) (a) Must use the definition of algebraic. What is $\cos \pi/4$? $1/\sqrt{2}$, so $\mathbf{Q}(\cos\pi/4)$ is $\mathbf{Q}(1/\sqrt{2})$.

Since $\cos\pi/4 = 1/\sqrt{2}$ and $(1/\sqrt{2})^2 - 1/2 = 0$, we have that $\cos\pi/4$ is a zero of the polynomial $t^2 - 1/2$ over $\mathbf{Q}$. $\mathbf{Q}(\cos\pi/4) : \mathbf{Q}$ *is* algebraic.

(b) π^2 seems unlikely to be algebraic over $\mathbf{Q}$! If it were, we'd get a polynomial equation for π.

If π^2 were a zero of $f(t) \in \mathbf{Q}[t]$, then $f(\pi^2) = 0$. Thus π would be a zero of $g(t) = f(t^2)$.

Since π is transcendental over $\mathbf{Q}$, $\mathbf{Q}(\pi^2) : \mathbf{Q}$ is *not* algebraic.

(ii) These are finite extensions, so test for being splitting fields.

(a) K contains *a* zero for $t^2 + t + 4$ so, since it's a quadratic, $t^2 + t + 4$ factorizes completely over K.

K contains a zero of $t^2 + t + 4$. Thus $t^2 + t + 4$ factorizes into linear factors over K. Now K is a splitting field for $t^2 + t + 4$ over $\mathbf{Z}_7$ and so $K : \mathbf{Z}_7$ is normal.

(b) $\sqrt[7]{2}$ is a zero of $t^7 - 2$, irreducible but not all zeros in $\mathbf{Q}(\sqrt[7]{2}, \sqrt{7})$.

$t^7 - 2$ has a zero in $\mathbf{Q}(\sqrt[7]{2}, \sqrt{7})$ but since $\mathbf{Q}(\sqrt[7]{2}, \sqrt{7}) \subseteq \mathbf{R}$, $t^7 - 2$ does not split over $\mathbf{Q}(\sqrt[7]{2}, \sqrt{7})$. Since $t^7 - 2$ is irreducible over $\mathbf{Q}$, the extension $\mathbf{Q}(\sqrt[7]{2}, \sqrt{7}) : \mathbf{Q}$ is *not* normal.

(c) Splitting field for $(t^7 - 2)(t^2 - 7)$.

Since $\mathbf{Q}(\sqrt[7]{2}, \sqrt{7}, \omega)$ is a splitting field for $(t^7 - 2)(t^2 - 7)$, $\mathbf{Q}(\sqrt[7]{2}, \sqrt{7}, \omega) : \mathbf{Q}$ *is* normal.

Question 8

(i) What is $K(\alpha^2)$? The smallest subfield such that ...

Since $K \subseteq K(\alpha)$ and $\alpha^2 \in K(\alpha)$, $K(\alpha^2)$ is, by definition, the smallest subfield of $K(\alpha)$ which contains K and α^2. In particular, $K(\alpha^2) \subseteq K(\alpha)$.

(ii) Could try to find the minpol of α^2 in terms of the minpol of α, but it gets messy. Easier to use 'finite implies algebraic'.

Since $K \subseteq K(\alpha^2) \subseteq K(\alpha)$, we have

$$[K(\alpha^2) : K] \le [K(\alpha) : K].$$

But $[K(\alpha) : K]$ is the degree of the minimum polynomial of α over K, which is n. So $K(\alpha^2) : K$ is finite, and hence algebraic, by Lemma 4.4. In particular, α^2 is algebraic over K.