Question 2

(i) What do I have to do … follow basic strategy for ideals. Closure under $+$, closure under additive inverses, possessing zero, closure under arbitrary multiplication.
Slight difficulty over elements.
Element of R/N is $r+N$ for $r \in R$.
Element of I/N is $x+N$ for $x \in I$.

Each step needs the definition of I/N, properties of the ideal I.

Closure under $+$ Suppose $x+N$, $y+N \in I/N$; then

$$(x+N)+(y+N)=(x+y)+N$$

(by definition of coset addition).

But $x, y \in I$ so $(x+y) \in I$ since I is an ideal. Hence $(x+y)+N \in I/N$.

Additive inverses Suppose $x+N \in I/N$; then

$$-(x+N)=(-x)+N.$$

But $x \in I$, so $-x \in I$, so $(-x)+N \in I/N$.

Zero Since I is an ideal, $0 \in I$. Hence $0+N=N \in I/N$. So N is the zero of R/N, and I/N does contain the zero of R/N.

Arbitrary multiplication Suppose $x+N \in I/N$ and $r+N \in R/N$. Then $(x+N)(r+N)=(xr)+N$. But $x \in I$, $r \in R$ so $xr \in I$ because I is an ideal. Thus $(xr)+N \in I/N$. Since R is commutative, so is R/N, and hence $(r+N)(x+N) \in I/N$.

(ii) What have fields to do with ideals? Fields have only trivial ideals so there's the germ of a contradiction argument here.

Suppose not: that is, suppose $I \neq R$, $I \neq N$. Then I/N is an ideal of R/N.

Because $I \neq R$, there is some coset $r+N$ not in I/N, so $I/N \neq R/N$. Because $I \neq N$, there is a coset $x+N \neq N$ in I/N. Hence I/N is non-trivial and not the whole ring R/N. But this contradicts Theorem 2.4.3, that the only ideals of the *field* R/N are R/N and the zero ideal. Thus either $I=N$ or $I=R$.

Question 3

(i) Basic strategy called for: closure under $+$, zero, closure under scalar multiplication.

Nearly forgot the subset requirement … bad trap to fall into.

Since the elements of W are of the form of those in K with $d=0$, we have $W \subseteq K$.

If $a+b\sqrt{2}+ci$, $a'+b'\sqrt{2}+c'i \in W$, then

$$(a+b\sqrt{2}+ci)+(a'+b'\sqrt{2}+c'i)$$
$$=(a+a')+(b+b')\sqrt{2}+(c+c')i \in W$$

because $\mathbf{Q}$ is closed under addition, so

$$a+a', b+b', c+c' \in \mathbf{Q}.$$

Since $0 \in \mathbf{Q}$, $0+0\sqrt{2}+0i=0 \in W$, so W contains the zero of K.
If $a+b\sqrt{2}+ci \in W$ and $\lambda \in \mathbf{Q}$, then

$$\lambda(a+b\sqrt{2}+ci)=(\lambda a)+(\lambda b)\sqrt{2}+(\lambda c)i \in W$$

because $\mathbf{Q}$ is closed under multiplication. Hence W is a vector subspace of K.

(ii) Need to prove spanning and linear independence.

$\{1, \sqrt{2}\}$ spans V by the definition of the elements of V as linear combinations of 1, $\sqrt{2}$ over $\mathbf{Q}$.

Suppose $a+b\sqrt{2}=0$. This means that $\sqrt{2}$ is rational unless $a=b=0$. Since $\sqrt{2}$ is irrational, we obtain $a=b=0$. Thus $\{1, \sqrt{2}\}$ is a linearly independent spanning set, hence a basis for V.

(iii) Need to prove spanning and linear independence. Spanning is OK because of definition of W, but independence looks harder.
Try a 'first principles' approach.
On the face of it we have $i=c^{-1}(-a-b\sqrt{2})$ which would make i real … but we'd better look at the possibility $c=0$.
Deal with $c \neq 0$, $c=0$ separately.

$\{1, \sqrt{2}, i\}$ spans W by the definition of the elements of W as linear combinations of 1, $\sqrt{2}$, i over $\mathbf{Q}$.

Suppose $a+b\sqrt{2}+ci=0$ for $a, b, c \in \mathbf{Q}$. If $c \neq 0$ then

$$i=c^{-1}(-a-b\sqrt{2})$$

which forces $i \in \mathbf{R}$; contradiction.

Thus $c=0$.

Then $a+b\sqrt{2}=0$. In part (ii) we showed that this implies that $a=b=0$. Thus $\{1, \sqrt{2}, i\}$ is a linearly independent spanning set, hence a basis.