Question 1

Let f and g be the polynomials in $\mathbf{Q}[t]$ defined by

$$f(t) = t^5 + 4t^4 + 3t^3 - 6t^2 - 18t - 12,$$
$$g(t) = t^2 + 3t + 2.$$

(i) Find a linear factor of f, and hence prove that f can be written in the form

$$f(t) = (t - \alpha)h(t)$$

with α in $\mathbf{Q}$ and h an irreducible polynomial in $\mathbf{Q}[t]$. [4]

(ii) Find an hcf for f and g, and express it in the form

$$af + bg$$

for suitable polynomials a and b in $\mathbf{Q}[t]$. [6]

Question 2

Let R be a commutative ring with a multiplicative identity (denoted by 1). Let I and N be ideals of R such that

$$N \subseteq I \subseteq R.$$

(i) Prove that I/N is an ideal of the quotient ring R/N. [6]

(ii) Suppose that R/N is a *field*. Prove that either $I = N$ or $I = R$. [4]

Question 3

Let

$$K = \{a + b\sqrt{2} + ci + di\sqrt{2} : a, b, c, d \in \mathbf{Q},\ i^2 = -1\},$$
$$W = \{a + b\sqrt{2} + ci : a, b, c \in \mathbf{Q},\ i^2 = -1\},$$

and let

$$V = \{a + b\sqrt{2} : a, b \in \mathbf{Q}\}.$$

You may assume that K and V are fields under addition and multiplication of complex numbers, that K is a vector space over $\mathbf{Q}$ and that $i \notin \mathbf{R}$.

(i) Prove that W is a vector subspace of K. [3]

(ii) Prove that $\{1, \sqrt{2}\}$ is a basis for V over $\mathbf{Q}$. [2]

(iii) Prove that $\{1, \sqrt{2}, i\}$ is a basis for W over $\mathbf{Q}$. [3]

(iv) Prove that W is not a subfield of K. [2]

Question 4

(i) Find all the elements in the conjugacy class of $\mathbf{S}_4$ containing the element $(12)(34)$. [3]

(ii) Find the order of the centralizer $C_{\mathbf{S}_4}((12)(34))$ of $(12)(34)$ in $\mathbf{S}_4$. [2]

(iii) Show that $C_{\mathbf{S}_4}((12)(34))$ is not cyclic. [2]

(iv) Find $C_{\mathbf{S}_4}((1234))$. [3]

Question 5

Suppose that H and N are normal subgroups of a group G, and that $HN = G$.

Prove that the quotient group $G/(H \cap N)$ is the internal direct product

$$(H/(H \cap N)) \times (N/(H \cap N)).$$ [10]