M433 Solutions to Specimen Examination Paper 1

These solutions are presented in two halves. The left-hand column gives an abbreviated version of the sort of thought processes that we'd go through in solving the problems; the right-hand column gives a fairly concise written solution. If you are one of the lucky people who can write polished solutions with no jottings, we envy you! If not, please don't be afraid to write down your ideas and intentions/strategies even if you don't manage to carry them out fully.

Question 1

(i) Linear factors? Try finding a zero. Try factors of constant term $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$. $f(1) = -28$, but $f(-1) = 0$. Divide to get h, either 'long' division or trial and error. Irreducibility: if in doubt, try Eisenstein. Everything divisible by 3 so …

Since

$$f(-1) = -1 + 4 - 3 - 6 + 18 - 12 = 0,$$

we have that $t + 1$ is a factor of f.

$f(t) = (t+1)(t^4 + 3t^3 - 6t - 12)$.

So $f(t) = (t+1)h(t)$ where $h(t) = t^4 + 3t^3 - 6t - 12$.

Since $3 \nmid 1$, $3|3$, $3|0$, $3|-6$, $3|-12$ and $3^2 \nmid -12$, h is irreducible over $\mathbf{Z}$ (and hence over $\mathbf{Q}$) by Eisenstein's criterion with $q = 3$.

(ii) Could use the Euclidean Algorithm directly *but* -1 is a zero of g too. Could use this.

Write $1 = (c)h + (d)(t+2)$ then multiply by $t+1$ to get $t + 1 = af + bg$. Apply Euclidean Algorithm to $t+2$, h.

$g(t) = (t+1)(t+2)$ so f, g have a common factor $t+1$.

h is irreducible and not equal to $t+2$, so f, g have $t+1$ as hcf.

$$\begin{array}{r|rrrrr}
 & t^3 & +\,t^2 & -\,2t & -\,2 & \\
\hline
t+2 & t^4 & +\,3t^3 & & -\,6t & -\,12 \\
 & t^4 & +\,2t^3 & & & \\
\cline{2-3}
 & & t^3 & & -\,6t & \\
 & & t^3 & +\,2t^2 & & \\
\cline{3-4}
 & & & -\,2t^2 & -\,6t & \\
 & & & -\,2t^2 & -\,4t & \\
\cline{4-5}
 & & & & -\,2t & -\,12 \\
 & & & & -\,2t & -\,4 \\
\cline{5-6}
 & & & & & -\,8
\end{array}$$

Thus $h = (t^3 + t^2 - 2t - 2)(t+2) - 8$.

Rearranging,

$$1 = -\tfrac{1}{8}h - \tfrac{1}{8}(t^3 + t^2 - 2t - 2)(t+2).$$

Hence $t + 1 = -\frac{1}{8}f - \frac{1}{8}(t^3 + t^2 - 2t - 2)g$.