(iii) 'Identify' means 'find a familiar group which is isomorphic to ...'. Easier to start by finding the order.

By the Fundamental Theorem of Galois Theory,

$|\Gamma(\mathbf{Q}(\alpha,\omega):\mathbf{Q})| = 6.$

There are lots of ways of identifying it as S_3 rather than C_6. One is to show it isn't abelian, which we can do by finding any two elements which don't commute. Try the two most 'obvious' automorphisms.

By the Automorphism Theorem, applied to $\mathbf{Q}(\alpha,\omega):\mathbf{Q}(\alpha)$, there is a $\mathbf{Q}(\alpha)$-automorphism σ of $\mathbf{Q}(\alpha,\omega)$ such that $\sigma(\omega)=\omega^2$.

Similarly, since the minimum polynomial of α over $\mathbf{Q}(\omega)$ is t^3-7, there is a $\mathbf{Q}(\omega)$-automorphism τ of $\mathbf{Q}(\alpha,\omega)$ such that $\tau(\alpha)=\omega\alpha$.

Now $\tau\sigma(\alpha)=\tau(\alpha)=\omega\alpha$ and $\sigma\tau(\alpha)=\sigma(\omega\alpha)=\sigma(\omega)\sigma(\alpha)=\omega^2\alpha$.

$\Gamma(\mathbf{Q}(\alpha,\omega):\mathbf{Q})$ contains both σ and τ, so it is not abelian, so it is isomorphic to S_3.

Question 12

(i) What does same *shape* mean? *Similar*. Need to turn this *geometric* condition into an *algebraic* one.

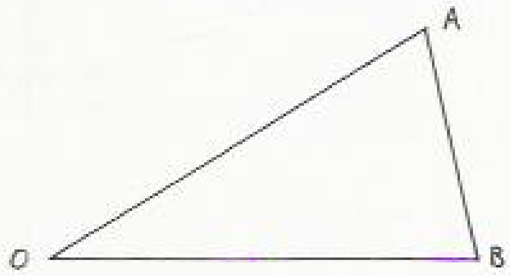
We may consider AB to have unit length. Let y be the length of BC. Then the large rectangle is $1\times y$ and the small rectangles are $y\times\frac{1}{2}$: for these to have the same shape we need $\dfrac{1}{y}=\dfrac{y}{1/2}$, i.e. $y=1/\sqrt{2}$.

Since $[\mathbf{Q}(1/\sqrt{2}):\mathbf{Q}]=2$, Theorem 17.3 shows that we can construct a line of length $1/\sqrt{2}$.

Length isn't enough: got to get the angles right too.

Since we can construct right-angles, we can construct lines BC and AD at right-angles to AB and with length $AB/\sqrt{2}$: then we can join CD, and this completes construction of the required rectangle.

(ii) We usually start with a *radius* of the polygon, not a *side*, so let's try to construct the centre.



Let O be the centre of the polygon. Then $\angle AOB = 2\pi/204$ and $\angle OAB=\angle OBA=(2\pi-2\pi/204)/2$.

Theorem 17.11 *must* be relevant!

$204=2^2\cdot 3\cdot 17$, and 3, 17 are Fermat primes, so, by Theorem 17.11, the angle $2\pi/204$ is constructible. Hence the angles $2\pi-2\pi/204$ and $(2\pi-2\pi/204)/2$ (by bisection) are constructible. By constructing angles of $(2\pi-2\pi/204)/2$ at A and at B, we can construct the point O.

Given O and the line segment OA, Theorem 17.11 now tells us that we can construct a regular 204-gon with centre O and one radius OA: this is the required polygon.

Question 13

The basic strategy available is to show that the Galois group of f is S_5, which is not soluble. Therefore we must locate the zeros of f, and apply Theorem 14.7. Locate changes of sign in f and look at f'.

First, irreducibility. By Eisenstein's criterion with $q=3$, f is irreducible over $\mathbf{Z}$ and hence over $\mathbf{Q}$. We have

$$f(-2)<0,\quad f(-1)>0,\quad f(0)<0,\quad f(2)>0.$$

Thus, by the Intermediate Value Theorem, f has at least three real zeros.

Now $f'(t)=5t^4-6$ which has two real zeros: $\pm\sqrt[4]{6/5}$. Hence, by Rolle's Theorem, f has at most 3 real zeros. Hence f has exactly three real zeros. Since f splits in $\mathbf{C}$, f has 2 non-real zeros.

By Theorem 14.7, since 5 is prime, the Galois group of f over $\mathbf{Q}$ is S_5. Since S_5 is not soluble, by Theorem 13.5, by Theorem 14.6 f is not soluble by radicals.