Question 1

Let m and p be the polynomials in $\mathbf{Q}[t]$ defined by

$$m(t) = t^5 + 2t + 6,$$
$$p(t) = t - 3.$$

(i) Prove that m is irreducible over $\mathbf{Q}$. [3]

(ii) Find a highest common factor of m and p, and express it in the form $um + vp$ where u and v are polynomials in $\mathbf{Q}[t]$. [5]

(iii) Write down the multiplicative inverse of $p + \langle m \rangle$ in $\mathbf{Q}[t]/\langle m \rangle$. [2]

Question 2

Let R be a commutative ring with a multiplicative identity (denoted by 1).

An element $p \in R$ is said to be prime if the quotient ring $R/\langle p \rangle$ is an integral domain. Prove that if p is a prime in R and $x, y \in R$, then

$$xy \in \langle p \rangle \quad \text{if and only if} \quad x \in \langle p \rangle \text{ or } y \in \langle p \rangle. \qquad [10]$$

Question 3

Let V be a finite-dimensional vector space over a field K.

Let $\phi : V \longrightarrow V$ be a linear transformation with the property that $\phi^2 = \phi$, i.e.

$$\phi(\phi(v)) = \phi(v) \quad \text{for all } v \text{ in } V.$$

$X \cap Y = \{v : \phi(v) = 0\} \cap \{\phi(v)\}$ ie soln to $w = \phi(v)$ so $\phi(w) = \phi^2(v)$, $0 = \phi(v) = w\}$

Let $\mathrm{Ker}(\phi) = X$ and $\mathrm{Im}(\phi) = Y$.

(i) Prove that $X \cap Y = \{0\}$. [4]

(ii) Prove that every element $v \in V$ can be written in the form

$$v = x + y \quad \text{where } x \in X \text{ and } y \in Y. \qquad [3]$$

(iii) Prove that if $v \in V$ and

$$v = x_1 + y_1 = x_2 + y_2 \quad \text{where } x_1, x_2 \in X \text{ and } y_1, y_2 \in Y,$$

then $x_1 = x_2$ and $y_1 = y_2$. [3]

Question 4

Let $\mathbf{V}$ be the normal subgroup of order 4 in $\mathbf{S}_4$; that is,

$$\mathbf{V} = \{1, (12)(34), (13)(24), (14)(23)\}.$$

(i) Write down three distinct subgroups H_1, H_2, H_3 of $\mathbf{S}_4$ such that

$$H_i \cong \mathbf{V}, \quad H_i \neq \mathbf{V} \qquad (i = 1, 2, 3). \qquad [3]$$

(ii) Find elements $g, h \in \mathbf{S}_4$ such that

$$g^{-1} H_1 g = H_2$$

and

$$h^{-1} H_2 h = H_3. \qquad [4]$$

[Note that $g^{-1}H_1 g$ is, by definition, $\{g^{-1}xg : x \in H_1\}$, and similarly for $h^{-1}H_2 h$.]

(iii) Does there exist an element $k \in \mathbf{S}_4$ such that $k^{-1}H_1 k = \mathbf{V}$?

Justify your answer. [3]

Question 5

Let G be a group and N a normal subgroup of G such that G/N is abelian. Let H be a subgroup of G such that $N \subseteq H$. Prove that H is a normal subgroup of G. [10]