Question 11

Let K be the field $\mathbf{Q}(\sqrt{5} + i\sqrt{5})$, and let G be the Galois group of $K : \mathbf{Q}$.

(i) Prove that $K = \mathbf{Q}(\sqrt{5}, i)$. [5]

(ii) Find G, specifying for each element of G its effect on $\sqrt{5}$ and on i. [5]

Question 12 (Unit 13)

Let p be a prime of the form $2r + 1$ where r is an odd positive integer. Let Σ be the subfield of $\mathbf{C}$ which is a splitting field for $t^p - 1$ over $\mathbf{Q}$. Let $L = \Sigma \cap \mathbf{R}$.

(i) Find the degrees $[\Sigma : \mathbf{Q}]$ and $[\Sigma : L]$, and hence $[L : \mathbf{Q}]$. [6]

(ii) Let $\alpha \in L$. Prove that the point $(\alpha, 0)$ is constructible using ruler and compasses from the points $(0,0)$ and $(1,0)$ if and only if $\alpha \in \mathbf{Q}$. [4]

Question 13 (Unit 14)

Let n be a positive integer and let $\varepsilon = e^{2\pi i/n}$.

(i) Show that the zeros of $t^n - 1$ in $\mathbf{C}$ are

$$\varepsilon^j, \quad j = 0, 1, \ldots, n-1.$$ [2]

(ii) Show that $\mathbf{Q}(\varepsilon)$ is a splitting field for $t^n - 1$ over $\mathbf{Q}$. [2]

(iii) Show that the Galois group $\Gamma(\mathbf{Q}(\varepsilon) : \mathbf{Q})$ is abelian. [3]

(iv) Prove that if M is any field such that

$$\mathbf{Q} \subseteq M \subseteq \mathbf{Q}(\varepsilon),$$

then the field extension $M : \mathbf{Q}$ is normal. [3]

Question 14 (Unit 15)

Let f be an irreducible quadratic in $\mathbf{Z}_p[t]$. Let L be a splitting field for f over $\mathbf{Z}_p$. Prove that every quadratic polynomial in $\mathbf{Z}_p[t]$ splits over L. [10]

Question 15 (Unit 16)

Let $L : K$ be a purely inseparable extension. Let K have characteristic p, and let α be an element of L with exponent e.

(i) Find the minimum polynomial m of α over K, and express m as a product of linear factors over L. [3]

(ii) Prove that $\Gamma(L : K) = 1$. [7]

[END OF QUESTION PAPER]

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