Question 6

For *each* of the following three properties, describe, with brief justification, a group G of order 24 possessing that property.

(i) G is abelian but not cyclic. [4]

(ii) G is metabelian but not abelian. [3]

(iii) G is soluble but not metabelian. [3]

Question 7

(i) For *each* of the following field extensions, state, with brief reasons, whether or not it is algebraic.

(a) $\mathbf{Q}(\pi):\mathbf{Q}$ [2]

(b) $\mathbf{Q}(\pi, \sqrt{2}):\mathbf{Q}(\pi)$ [2]

(ii) For *each* of the following finite field extensions, state, with brief reasons, whether or not it is normal.

(a) $\mathbf{Q}(\sqrt[3]{5}):\mathbf{Q}$ [2]

(b) $\mathbf{Q}(\omega, \sqrt[3]{5}):\mathbf{Q}$, where $\omega^3 = 1$, $\omega \neq 1$. [2]

(c) $K:\mathbf{Z}_3$, where $K = \mathbf{Z}_3[t]/\langle t^2+1\rangle$. [2]

where $d(x)=0$
$X \cap Y = \{0\}$

Question 8

Let K and L be fields with $K \subseteq L$. Let $\alpha \in L$ and let the minimum polynomial of α have degree n over K.

(i) Prove that $K(\alpha^2) \subseteq K(\alpha)$. [1]

(ii) Prove that α^2 is algebraic over K. [2]

(iii) Suppose that n is *odd*. Prove that $K(\alpha^2) = K(\alpha)$. [5]

(iv) Give an example with $K = \mathbf{Q}$ such that K is a proper subset of $K(\alpha^2)$ and $K(\alpha^2)$ is a proper subset of $K(\alpha)$. [2]

Question 9

Let $N:K$ be a finite, normal, separable field extension, and let G be the Galois group $\Gamma(N:K)$ of $N:K$. Let H_1 and H_2 be subgroups of G such that H_1 is a normal subgroup of H_2. Let N_i be the fixed field $H_i^\dagger$ of H_i (for $i = 1, 2$).

(i) Prove that $N:N_2$ is finite, separable and normal. [6]

(ii) Prove that $N_1:N_2$ is normal. [2]

(iii) Prove that $\Gamma(N_1:N_2) \cong H_2/H_1$. [2]

Question 10

Let $K(\alpha):K$ be a simple, algebraic extension, and let f be the minimum polynomial of α over K. Let G be the Galois group $\Gamma(K(\alpha):K)$ of $K(\alpha):K$.

(i) Prove that the order of G is equal to the number of distinct zeros of f in $K(\alpha)$. [4]

(ii) Hence prove that the order of G is equal to the field extension degree $[K(\alpha):K]$ if and only if $K(\alpha):K$ is normal and α is separable over K. [6]