Question 6

(i) For each of the following groups state, with brief reasons, whether or not it is soluble.

(a) C_6 [1]

(b) A_4 [2]

(c) $\mathsf{C}_6 \times \mathsf{S}_6$ [3]

(ii) For each of the following two properties describe, with a brief justification, a group G of order 50 possessing that property.

(a) G is abelian but not cyclic. [2]

(b) G is metabelian but not abelian. [2]

Question 7

For each of the following field extensions state, with brief reasons, whether or not it is algebraic.

For each of the extensions which is algebraic state, with brief reasons, whether or not it is a normal field extension.

(i) $\mathbf{Q}(\sqrt[3]{7}+2) : \mathbf{Q}$ [4]

(ii) $\mathbf{Q}(\pi^2) : \mathbf{Q}$ [3]

(iii) $\mathsf{F}_4 : \mathsf{Z}_2$. [3]

Question 8

Let K and L be fields with $K \subset L$. Let α be an element of L that is algebraic over K with minimum polynomial of degree n over K.

(i) Prove that $K(\alpha^2) \subset K(\alpha)$. [2]

(ii) Prove that α^2 is algebraic over K. [2]

(iii) Suppose that n is *odd*. Prove that $K(\alpha^2) = K(\alpha)$. [4]

(iv) Give an example with $K = \mathbf{Q}$ such that K, $K(\alpha)$ and $K(\alpha^2)$ are distinct fields. [2]

Question 9

Let K be the field $\mathbf{Q}(\xi)$, where $\xi = e^{2\pi i/7}$, and let G be the Galois group $\Gamma(K : \mathbf{Q})$.

(i) Find G, specifying for each element of G its effect on ξ. [5]

(ii) How many subfields has K? Justify your answer. [5]

Question 10

Let $p(t)$ and $q(t)$ be two distinct irreducible quadratic polynomials in $\mathbf{Q}[t]$, and let K be the splitting field of the polynomial $p(t)q(t)$ over $\mathbf{Q}$. Let G be the Galois group $\Gamma(K : \mathbf{Q})$.

(i) What are the possible orders of G? [3]

(ii) For each possible order of G, identify G, justifying your answer. [3]

(iii) For each of the possible groups G, give an example of *distinct* polynomials $p(t)$ and $q(t)$ producing the group, justifying your answer. [4]