Question 11

Let K be the subfield of $\mathbf{C}$ which is the splitting field of the polynomial $t^4 - 25$ over $\mathbf{Q}$, and let G be the Galois group $\Gamma(K : \mathbf{Q})$.

(i) Find α and $\beta \in \mathbf{C}$ such that $K = \mathbf{Q}(\alpha, \beta)$. [3]

(ii) Prove that G is isomorphic to $\mathbf{V}$. [7]

Question 12 (Unit 13)

Determine whether or not the following ruler and compass constructions are possible, giving a justification in each case.

(i) Construction of a regular 680-gon. [3]

(ii) Construction of an acute angle α such that $\tan\alpha = \dfrac{3}{\sqrt[4]{5}}$. [4]

(iii) Construction of an acute angle β such that $\tan\beta = \dfrac{4}{\sqrt[3]{5}}$. [3]

Question 13 (Unit 14)

For each of the following polynomials in $\mathbf{Q}[t]$, determine whether or not it is soluble by radicals, giving a justification in each case.

(i) $t^5 - 14t + 7$ [7]

(ii) $t^5 + 6t + 7$ [3]

Question 14 (Unit 15)

Let p be a prime, let f be an irreducible polynomial of degree 10 in $\mathbf{Z}_p[t]$ and let L be a splitting field for f over $\mathbf{Z}_p$.

(i) Prove that every irreducible polynomial of degree 10 in $\mathbf{Z}_p[t]$ splits over L. [4]

(ii) Prove that every irreducible polynomial of degree 5 in $\mathbf{Z}_p[t]$ splits over L. [3]

(iii) State whether every polynomial of degree 4 in $\mathbf{Z}_p[t]$ splits over L, justifying your answer. [3]

Question 15 (Unit 16)

(i) For each of the following field extensions $L : K$ state, with brief reasons, whether or not $L : K$ is a simple extension. If $L : K$ is a simple extension write down an element α of L such that $L = K(\alpha)$.

(a) $\mathbf{Q}(i, \sqrt{5}) : \mathbf{Q}$ [3]

(b) $\mathbf{Z}_5(u, v) : \mathbf{Z}_5(u^5, v^5)$, where the extensions $\mathbf{Z}_5(u) : \mathbf{Z}_5$ and $\mathbf{Z}_5(u, v) : \mathbf{Z}_5(u)$ are both simple transcendental extensions. [2]

(ii) Let K be an ordered field.

(a) Prove that there is a field L containing K such that $[L : K] = 2$. [3]

(b) Is the field L in part (ii)(a) necessarily algebraically closed? Justify your answer with either a brief proof or a counter-example as appropriate. [2]

[END OF QUESTION PAPER]