Question 1

(i) (a) Find integers m and n such that

$$25m + 36n = 1.$$ [5]

(b) Write down the multiplicative inverse of 25 in the ring $\mathbf{Z}_{36}$. [1]

(ii) State whether or not $\mathbf{Z}_{36}$ is a field and justify your answer. [4]

Question 2

For each of the interpretations of $(*)$ given below, state whether the following is true:

if a ring R satisfies $(*)$ and I is an ideal of R, then the quotient ring R/I also satisfies $(*)$.

Give either a brief proof or a counterexample in each case.

(i) $(*)$ means *is commutative.* [2]

(ii) $(*)$ means *has a multiplicative identity.* [2]

(iii) $(*)$ means *is an integral domain.* [3]

(iv) $(*)$ means *is a field.* [3]

Question 3

Let L be the field $\mathbf{Q}(\sqrt[4]{5}, i)$.
You may assume that L is a vector space over $\mathbf{Q}$.

(i) Find a basis for L over $\mathbf{Q}$, justifying your answer. [5]

(ii) Let $K = \{a + b\sqrt[4]{5} + c\sqrt{5} : a, b, c \in \mathbf{Q}\}$.

(a) Is K a vector subspace of L over $\mathbf{Q}$? Justify your answer. [2]

(b) Is K a subring of L? Justify your answer. [3]

Question 4

(i) Let θ be the permutation $(1\,2\,3)(4\,5)(6)$ in the group $\mathbf{S}_6$.

(a) Write down the order of the element θ in $\mathbf{S}_6$. [1]

(b) Write down the sign of the permutation θ. [1]

(ii) Prove that every odd permutation in $\mathbf{S}_n$ has even order in $\mathbf{S}_n$. [6]

(iii) Give an example of each of the following:

(a) an even permutation in $\mathbf{S}_6$ that has even order in $\mathbf{S}_6$; [1]

(b) an even permutation in $\mathbf{S}_6$ that has odd order in $\mathbf{S}_6$. [1]

Question 5

A *maximal* normal subgroup N of a group G is a proper normal subgroup of G with the property that for any normal subgroup N' of G such that $N \subseteq N' \subseteq G$, then it follows that $N' = N$ or $N' = G$.

Prove that a normal subgroup N of a group G is a maximal normal subgroup of G if and only if the quotient group G/N is a simple group with more than one element. [10]