Question 6

(i) For each of the following groups state, with brief reasons, whether or not it is simple.

(a) C_7 [2]

(b) S_4 [2]

(ii) For each of the following groups state, with brief reasons, whether or not it is soluble.

(a) C_7 [2]

(b) A_4 [2]

(c) $S_5 \times C_7$ [2]

Question 7

(i) For *each* of the following field extensions state, with brief reasons, whether or not it is algebraic.

(a) $\mathbf{R} : \mathbf{Q}$ [2]

(b) $\mathbf{C} : \mathbf{R}$ [2]

(ii) For *each* of the following field extensions state, with brief reasons, whether or not it is normal.

(a) $\mathbf{Q}(\sqrt{3}) : \mathbf{Q}$ [2]

(b) $\mathbf{Q}(\alpha) : \mathbf{Q}$, where $\alpha^5 = 1$, $\alpha \neq 1$ [2]

(c) $\mathbf{Z}_{29}(u) : \mathbf{Z}_{29}(u^2)$, where u is transcendental over $\mathbf{Z}_{29}$ [2]

Question 8

Let K, L be fields with $K \subseteq L$, and let R be a *ring* such that $K \subseteq R \subseteq L$.

(i) Suppose that $[L : K]$ is finite and that r is an element of R. Prove that r is algebraic over K. [2]

(ii) Prove that if $[L : K]$ is finite then R is a subfield of L. $0 = \sum a_n r^n$ $\quad -a_0 = \sum_{n=a} a_n r^n$ $\quad -\frac{a_0}{r} = \sum a_n r^{n-1}$

$r = \sum a_n \pi^n$ (iii) Find an example to show that R need not be a subfield of L when $[L : K]$ is infinite. [3]

$\pi^{-1} \in R \Rightarrow \frac{1}{\pi} = \sum a_n \pi^n \Rightarrow \sum a_n \pi^{n+1} - 1 = 0$ but π transcendental $\Rightarrow$ no polynomial exists

Question 9

Let K be the field $\mathbf{Q}(\psi)$ where $\psi = e^{2i\pi/11}$. Let G be the Galois group $\Gamma(K : \mathbf{Q})$.

(i) Find G, fully justifying you answer and specifying, for each element of G, its effect on ψ. [7]

(ii) Prove that K has exactly four subfields. [3]

Question 10

Let $p(t)$ and $q(t)$ be irreducible quadratic polynomials in $\mathbf{Q}[t]$, and let K be a splitting field for the polynomial $p(t)q(t)$ over $\mathbf{Q}$. Let G be the Galois group $\Gamma(K : \mathbf{Q})$.

(i) Prove that G is isomorphic either to C_2 or to V. [7]

(ii) Find an example of two such polynomials $p(t)$ and $q(t)$, both monic with $p(t) \neq q(t)$, for which this group G is isomorphic to C_2. [3]