Question 11

Let K be the subfield of $\mathbf{C}$ which is the splitting field of the polynomial t^7+4 in $\mathbf{Q}[t]$ which you may assume to be irreducible, and let G be the Galois group $\Gamma(K:\mathbf{Q})$.

(i) Find $\alpha, \beta \in \mathbf{C}$ such that $K = \mathbf{Q}(\alpha, \beta)$. [3]

(ii) Find the degree $[K:\mathbf{Q}]$. [3]

(iii) Show that the group G has a normal subgroup N such that both N and G/N are cyclic; identify the groups N and G/N. [4]

Question 12

Determine whether or not the following ruler and compasses constructions are possible. Give a justification in each case.

(i) Construction of an equilateral triangle equal in area to a given square. [7]

(ii) Construction of a regular 440-gon. [3]

Question 13

Prove that each of the following polynomials in $\mathbf{Q}[t]$ is soluble by radicals over $\mathbf{Q}$.

(i) $t^3 - 2t + 7$ [3]

(ii) $t^6 + 9$ [3]

(iii) $t^6 - 2t^3 + 17$ [4]

Question 14

Let K be the field $\mathrm{GF}(2^6)$, let the cyclic group H be its multiplicative group, and let x be a generator of H.

(i) Write down the number of elements in each subfield of K. [2]

(ii) For each of the subfields identified in part (i), describe its multiplicative group in terms of x. [4]

(iii) Let 0 be the additive identity of K. Find an example of a subgroup T of H such that $T \cup \{0\}$ is *not* a subfield of K, and explain why it is not. [4]

Question 15

Let K be an ordered field.

(i) Prove that there is a field L containing K such that $[L:K] = 2$. [6]

(ii) Is L necessarily algebraically closed? Justify your answer with a brief proof or counterexample, as appropriate. [4]

[END OF QUESTION PAPER]