Question 1

(i) (a) Find integers m and n such that

$$18m + 25n = 1$$ [5]

(b) Write down the multiplicative inverse of 18 in $\mathbf{Z}_{25}$. [1]

(ii) (a) State whether or not $\mathbf{Z}_{25}$ is a field. [2]

(b) Justify your answer to part (ii)(a) by either giving a short proof or giving a field axiom which $\mathbf{Z}_{25}$ fails to satisfy, with an explicit counterexample illustrating the failure. [2]

Question 2

Let R be a commutative ring with a multiplicative identity (denoted by 1). Let I and N be ideals of R such that

$$N \subseteq I \subseteq R.$$

(i) Prove that I/N is an ideal of the quotient ring R/N. [6]

(ii) Suppose that R/N is a *field*. Prove that either $I = N$ or $I = R$. [4]

Question 3

Let K, W and V be the following subsets of the real numbers.

$$K = \{a + b\sqrt{3} + c\sqrt{5} + d\sqrt{15} : a, b, c, d \in \mathbf{Q}\}$$
$$W = \{a + b\sqrt{3} + c\sqrt{5} : a, b, c \in \mathbf{Q}\}$$
$$V = \{a + b\sqrt{3} : a, b \in \mathbf{Q}\}$$

You may assume that K and V are fields under addition and multiplication of real numbers, that K, W and V are vector spaces over $\mathbf{Q}$ and that neither $\sqrt{3}$ nor $\sqrt{5}$ is in $\mathbf{Q}$.

(i) Prove that $\{1, \sqrt{3}\}$ is a basis for V over $\mathbf{Q}$. [2]

(ii) Prove that $\{1, \sqrt{3}, \sqrt{5}\}$ is a basis for W over $\mathbf{Q}$. [5]

(iii) Prove that W is not a subfield of K. [3]

Question 4

(i) Write down *three* distinct subgroups H_1, H_2 and H_3 of the symmetric group $\mathbf{S}_6$ such that

$$H_i \cong \mathbf{C}_3 \quad (i = 1, 2, 3),$$

and such that H_1 is conjugate to H_2 in $\mathbf{S}_6$ but H_1 is not conjugate to H_3 in $\mathbf{S}_6$. [3]

(ii) Find an element g of $\mathbf{S}_6$ such that h does *not* commute with every element of H_1, but

$$g^{-1}H_1g = H_1.$$ [2]

(iii) Find an element h of $\mathbf{S}_6$ such that

$$h^{-1}H_1h = H_2.$$ [2]

(iv) Prove that H_1 and H_3 are *not* conjugate in $\mathbf{S}_6$. [3]

Question 5

Let G be a group and let N be a normal subgroup of G such that the quotient group G/N is abelian. Let H be a subgroup of G such that $N \subseteq H$.

Prove that H is a normal subgroup of G and that G/H is an abelian group. [10]