Question 10

Let K be a field, f be an irreducible polynomial of degree 2 in $K[t]$, and L be the splitting field for f over K. Let α and β be the zeros of f in L, and G be the Galois group $\Gamma(L : K)$.

(i) Suppose that the polynomial f is separable.

(a) Find the elements of G, specifying the effect of each automorphism on α.

(b) Write down the order of G.

(c) Find the fixed field of G [6]

(ii) repeat part (i) in the case when the polynomial f is inseparable. [4]

Question 11

Let K be the subfield of $\mathbf{C}$ which is the splitting field of the irreducible polynomial $f(t) = t^4 + 9$ in $\mathbf{Q}[t]$, let $\alpha = \sqrt{3}e^{i\pi/4}$ and let G be the Galois group $\Gamma(K : \mathbf{Q})$.

(i) Prove that α is a solution of the equation $f(t) = 0$. [3]

(ii) Show that the other solutions of $f(t) = 0$ are $\frac{1}{3}\alpha^3$, $\frac{1}{9}\alpha^5$ and $\frac{1}{27}\alpha^7$ and hence that $K = \mathbf{Q}(\alpha)$. [4]

(iii) Prove that the group $G \cong \mathbf{V}$.

Question 12 (Unit 13)

Determine whether or not the following ruler and compass constructions are possible. Give a justification in each case.

(i) Construction of a rectangle, twice as long as it is wide, equal in area to a given circle. [6]

(ii) Construction of the angle $4\pi/105$. [4]

Question 13 (Unit 14)

Of the two following polynomials in $\mathbf{Q}[t]$, one is soluble by radicals and the other is not. Decide which is soluble and which is not, giving a justification in each case.

(i) $t^3 - 51t^2 + 3t + 2$ [3]

(ii) $t^5 - 14t + 7$ [7]

Question 14 (Unit 15)

Let K be the field $\mathrm{GF}(3^4)$, let the cyclic group H be its multiplicative group and let u be a generator of H.

(i) Write down the number of elements in each subfield of K. [2]

(ii) For each of the subfields identified in part (i) describe its multiplicative group in terms of u. [4]

(iii) Describe the elements of the Galois group of $\mathrm{GF}(3^4)$ over $\mathrm{GF}(3)$ in terms of their effect on u. [4]