Attempt as many questions as you wish. Full marks may be obtained for complete *solutions to* TEN *questions.*

Question 1

(i) Find a highest common factor d of 1988 and -5041 and find integers u and v such that $1988u - 5041v = d$. [4]

(ii) Write down an element of $\mathbf{Z}_{1988}$ (other than 1 and 1987) which has a multiplicative inverse. [2]

(iii) Write down an element of $\mathbf{Z}_{1988}$ (other than 0) which has no multiplicative inverse. [2]

(iv) Decide whether the ring $\mathbf{Z}_{1988}$ has ideals other than $\{0\}$ and $\mathbf{Z}_{1988}$ itself, giving reasons for your answer. [2]

Question 2

Let R be a ring in which every element r satisfies the equation $r^2 = r$.

(i) By expanding $(r + r)^2$, prove that every element r of R satisfies $r = -r$. [3]

(ii) By expanding $(x + y)^2$, prove that R is commutative. [3]

(iii) Prove that if R is an integral domain then R has at most two elements. [4]

Question 3

Let K, W and V be the following subsets of complex numbers, where $i^2 = -1$.

$$K = \left\{a + b\sqrt{2} + ci + di\sqrt{2} : a, b, c, d \in \mathbf{Q}, i^2 = -1\right\},$$

$$W = \left\{a + b\sqrt{2} + ci : a, b, c \in \mathbf{Q}, i^2 = -1\right\},$$

$$V = \left\{a + b\sqrt{2} : a, b \in \mathbf{Q}\right\}.$$

You may assume that K and V are fields under addition and multiplication of complex numbers, that K and V are vector spaces over $\mathbf{Q}$ and that $i \notin \mathbf{R}$.

(i) Prove that W is a vector subspace of K. [4]

(ii) Prove that $\{1, \sqrt{2}\}$ is a basis for V over $\mathbf{Q}$. [2]

(iii) Prove that $\{1, \sqrt{2}, i\}$ is a basis for W over $\mathbf{Q}$. [2]

(iv) Prove that W is not a subfield of K. [2]

Question 4

Let $\mathbf{V}$ be the following subgroup of order 4 in $\mathbf{S}_4$:

$$\mathbf{V} = \{1, (12)(34), (13)(24), (14)(23)\}.$$

(i) Write down three distinct subgroups H_1, H_2 and H_3 of $\mathbf{S}_4$ such that

$$H_i \cong \mathbf{V}, \quad H_i \neq \mathbf{V} \qquad (i = 1, 2, 3).$$ [3]

(ii) Find elements $g, h \in \mathbf{S}_4$ such that

$$g^{-1}H_1g = H_2$$

and

$$h^{-1}H_1h = H_3.$$ [4]

[Note that, by definition, $g^{-1}H_1g = \{g^{-1}xg : x \in H_1\}$ and similarly for $h^{-1}H_2h$.]

(iii) Does there exist an element $k \in \mathbf{S}_4$ such that $k^{-1}H_1k = \mathbf{V}$? Justify your answer. [3]