Question 5

(i) Let G be a group and let Z be the centre of G. Suppose further that the quotient group G/Z is cyclic. Prove that the group G is abelian. [7]

(ii) Give an example of a non-abelian group G such that the quotient group of G by its centre is abelian. [3]

Question 6

Let the group G be the direct product $H \times K$.
Consider each statement below, and state whether it is true or false, giving a brief proof or counter-example, as appropriate.

(i) If H and K are cyclic then G is cyclic. [3]

(ii) If H and K are abelian then G is abelian. [3]

(iii) If H and K are simple then G is simple. [4]

Question 7

(i) For each of the following field extensions state, with brief reasons, whether or not it is algebraic.

(a) $\mathbf{Q}(\pi):\mathbf{Q}$ [2]

(b) $\mathbf{Q}(\pi, \sqrt{2}):\mathbf{Q}(\pi)$. [2]

(ii) For each of the following finite field extensions state, with brief reasons, whether or not it is normal.

(a) $\mathbf{Q}(\sqrt[3]{5}):\mathbf{Q}$ [2]

(b) $\mathbf{Q}(\omega, \sqrt[3]{5}):\mathbf{Q}$ where $\omega^3 = 1$, $\omega \neq 1$. [2]

(c) $\mathbb{F}_4:\mathbf{Z}_2$. [2]

Question 8

Let K and L be fields with $K \subseteq L$. Let α be a element of L that is algebraic over K with minimum polynomial of degree n over K.

(i) Prove that $K(\alpha^2) \subseteq K(\alpha)$. [2]

(ii) Prove that α^2 is algebraic over K. [2]

(iii) Suppose that n is *odd*. Prove that $K(\alpha^2) = K(\alpha)$. [4]

(iv) Give an example with $K = \mathbf{Q}$ such that K is a proper subset of $K(\alpha^2)$ and $K(\alpha^2)$ is a proper subset of $K(\alpha)$. [2]

Question 9

Let K be the field $\mathbf{Q}(\xi)$, where $\xi = e^{2i\pi/7}$. Let G be the Galois group $\Gamma(K:\mathbf{Q})$.

(i) Find G, specifying for each element of G its effect on ξ. [5]

(ii) Prove that K has only four subfields. [5]