Loughborough University

COMPUTATIONAL FLUID DYNAMICS - 1
(09TTC001)

January 2010 — 2 Hours

Answer **THREE** questions

Only University-approved calculators are permitted

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1. a) i) Simplify the following continuity and momentum equations (Cartesian form) for steady, constant density, inviscid, two-dimensional flow.

Continuity: $$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)=0$$

x – momentum:

$$\frac{\partial(\rho u)}{\partial t}+\frac{\partial}{\partial x}(\rho u^2)+\frac{\partial}{\partial y}(\rho uv)+\frac{\partial}{\partial z}(\rho uw)=\frac{-\partial p}{\partial x}+$$
$$\frac{\partial}{\partial x}(\tau_{xx})+\frac{\partial}{\partial y}(\tau_{yx})+\frac{\partial}{\partial z}(\tau_{zx})$$

y – momentum:

$$\frac{\partial(\rho v)}{\partial t}+\frac{\partial}{\partial x}(\rho uv)+\frac{\partial}{\partial y}(\rho v^2)+\frac{\partial}{\partial z}(\rho vw)=\frac{-\partial p}{\partial y}+$$
$$\frac{\partial}{\partial x}(\tau_{xy})+\frac{\partial}{\partial y}(\tau_{yy})+\frac{\partial}{\partial z}(\tau_{zy})$$

[6 marks]

ii) Use these simplified equations to show that in such flow the total pressure P_t, defined as

$$P_t=p+\frac{1}{2}\rho\left(u^2+v^2\right)$$

is uniform everywhere if the flow is irrotational.

[8 marks]

b) Express the continuity and x-momentum equations, given in part (a) above, in conservative tensor notation form for incompressible flows.

[6 marks]

2. a) Write a brief note on the following mathematical types used for classification of pde's:

- Parabolic
- Hyperbolic
- Elliptic

Use graphs to support your answer.

[6 marks]

b) A non-linear 1D wave-motion in a flowing fluid is governed by the following equations:

$$\rho\frac{\partial u}{\partial t}+\rho u\frac{\partial u}{\partial x}=-\frac{\partial P}{\partial x}$$

$$\rho a^2\frac{\partial u}{\partial x}+\frac{\partial P}{\partial t}+u\frac{\partial P}{\partial x}=0$$

where "a" is the speed of sound.

i) Determine the characteristic directions in (x,t) space. [8 marks]

ii) Classify this system of equations as parabolic, hyperbolic or elliptic. [2 marks]

iii) Draw the domain of dependence/region of influence diagram. [4 marks]

3. a) Briefly describe each of the following criteria that can be used to judge the performance of various differencing schemes applied to partial differential equations of interest in CFD:

- Consistency
- Stability
- Convergence
- Numerical Accuracy

[2 marks each]

b) Deduce, using finite-volume methods, the algebraic equation corresponding to the following steady-state, first-order pde:

$$\frac{\partial(\rho U\varphi)}{\partial x}+\frac{\partial(\rho V\varphi)}{\partial y}=0$$

Assume the two velocity components are everywhere uniform $U=U_o$ and $V = V_o$ (both positive) and use the following interpolating practice for the cell face values of φ:

Second-order upwind interpolation:

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i.e. $\varphi_{i-\frac{1}{2}} = \varphi_{i-1} + \left.\frac{\partial\varphi}{\partial x}\right|_{i-\frac{1}{2}} \cdot \frac{\Delta x}{2}$ and use <u>upwind</u> approximation for $\left.\frac{\partial\varphi}{\partial x}\right|_{i-\frac{1}{2}}$

leading to: $\varphi_{i-\frac{1}{2}} = \left(\frac{3}{2}\varphi_{i-1} - \frac{1}{2}\varphi_{i-2}\right)$

Express the algebraic equations in the form:

$$a_P\varphi_P = a_N\varphi_N + a_S\varphi_S + a_E\varphi_E + a_W\varphi_W + S$$

Under what condition is the coefficient matrix diagonally dominant?

[12 marks]

4. a) Show that the conservation equation for mechanical energy given below may be derived from the momentum equations :

$$\rho\frac{DV^2}{Dt} = -u\frac{\partial p}{\partial x} - v\frac{\partial p}{\partial y} - w\frac{\partial p}{\partial z} + u\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)$$
$$+ v\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) + w\left(\frac{\partial \tau_{xz}}{\partial} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right)$$

where: $V^2 = \frac{1}{2}\left(u^2 + v^2 + w^2\right)$

[12 marks]

b) Burger's equation is defined as relevant to constant density, inviscid, one-dimensional (spatial) but unsteady flow.

i) List the simplifications which result from these assumptions.
ii) Reduce the full version of the momentum equations to this form.
iii) What are the properties of this equation which make it suitable as a model problem for development of numerical methods?
iv) How could this equation be linearised?

[2 marks each]

[Note: You may use the equations presented in question 1.(a) above to answer parts a&b of this question]

5. a) State the difference between the following:

- Explicit and implicit CFD schemes.
- Finite difference and finite volume descretization methods.
- Point- and Line-Gauss-Seidel iterative methods.

[4 marks each]

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b) Illustrate the application of the Alternating-Direction-Implicit (ADI) stability method to obtain the finite difference equation for the following unsteady 2-dimensional diffusion equation:

$$\frac{\partial T}{\partial t} - \alpha_x \frac{\partial^2 T}{\partial x^2} - \alpha_y \frac{\partial^2 T}{\partial y^2} = 0$$

[8 marks]

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