This paper is not to be removed from the Examination Halls

UNIVERSITY OF LONDON **279 0095 ZA**

BSc degrees and Diplomas for Graduates in Economics, Management, Finance and the Social Sciences, the Diploma in Economics and Access Route for Students in the External Programme

Further Mathematics for Economists

Friday, 9 June 2006 : 10.00am to 1.00pm

Candidates should answer **EIGHT** of the following **TEN** questions: **SIX** from Section A (60 marks in total) and **TWO** from Section B (20 marks each). **Candidates are strongly advised to divide their time accordingly.**

Throughout, **R** denotes the set of real numbers.

Graph paper is provided. If used, it must be fastened securely inside the answer book.

A hand held calculator may be used when answering questions on this paper but it must not be pre-programmed or able to display graphics, text or algebraic equations. The make and type of machine must be stated clearly on the front cover of the answer book.

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SECTION A

Answer all **six** questions from this section (60 marks in total).

1.(a) Show that the set $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ of vectors in $\mathbb{R}^3$ is linearly dependent, where

$$\mathbf{x}_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 8 \\ -7 \\ 0 \end{pmatrix}.$$

Express $\mathbf{x}_3$ as a linear combination of $\mathbf{x}_1$ and $\mathbf{x}_2$.

(b) Suppose that V is a vector space. What does it mean to say that the subset W of V is a 'subspace' of V?

Suppose that V is the vector space of all 2×2 matrices with real numbers as entries, with the usual addition and scalar multiplication operations. (You may assume that this is indeed a vector space.) Show that the set of all diagonal 2×2 matrices is a subspace of V. Show, however, that the sets

$$W = \left\{ \begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix} : a, b \in \mathbb{R} \right\} \quad \text{and} \quad U = \left\{ \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix} : a, b \in \mathbb{R} \right\}$$

are *not* subspaces of V.

2. What does it mean to say that a basis of a vector space is an 'orthonormal' basis?

Find an orthonormal basis for the subspace W of $\mathbb{R}^3$ given by

$$W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x - 2y + 3z = 0 \right\}.$$

Extend this to an orthonormal basis of $\mathbb{R}^3$.

3. What, precisely, does it mean to say that a sequence (x_n) of real numbers converges to the real number L as n tends to infinity? Prove, using this precise definition that, if

$$x_n = \frac{n^2 + n + 1}{2n^2 - n + 5},$$

then $x_n \to 1/2$ as $n \to \infty$.

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4. What, precisely, does it mean to say that a function $f : \mathbb{R} \to \mathbb{R}$ has limit L as $x \to a$, where $L, a \in \mathbb{R}$?

Suppose that the functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are such that $f(x) \to L$ and $g(x) \to M$ as $x \to a$. Prove, using the formal definition of limit, that the function h given by $h(x) = f(x)g(x)$ has the property that $h(x) \to LM$ as $x \to a$.

5. Find the value and optimal mixed strategies of the matrix game with pay-off matrix

$$\begin{pmatrix} 5 & 3 \\ 2 & 4 \end{pmatrix},$$

explaining your method.

6. By solving its dual, find the solution of the following linear programming problem:

$$\text{minimise} \quad 8x + 3y + 8z$$

subject to

$$\begin{aligned} x + y + 3z &\geq 6 \\ 4x + y + 2z &\geq 5 \\ x, y, z &\geq 0. \end{aligned}$$

SECTION B

Answer **two** questions from this section (20 marks each).

7.(a) The function g is given by

$$g(x, y, z) = x^2 + 5y^2 + z^2 + 2xy + Ayz,$$

where A is a positive constant. Show that g is convex if $A < 4$ but that it is not convex if $A > 4$.

(b) Suppose that

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Show that $\mathbf{v}$ is an eigenvector of A, and find the corresponding eigenvalue. Find the remaining eigenvalues of A and, for each of these eigenvalues, find a corresponding eigenvector.

Find an orthogonal matrix P and a diagonal matrix D such that $P^T AP = D$.

Let f denote the quadratic form

$$f(x, y, z) = x^2 + 2y^2 + z^2 + 2xy + 4xz + 2yz.$$

Show that it is possible to write f in the form

$$f(x, y, z) = \lambda_1 X^2 + \lambda_2 Y^2 + \lambda_3 Z^2,$$

for some numbers $\lambda_1, \lambda_2, \lambda_3$, where x, y, z are linear combinations of X, Y, Z. (You should state explicitly how x, y, z may be written in terms of X, Y, Z.)

Hence, or otherwise, find a vector $\mathbf{x}$ for which $f(x, y, z) = -4$.

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8.(a) If $\mathbf{x} = P\mathbf{z}$ where P is an orthogonal matrix, show that $||\mathbf{x}|| = ||\mathbf{z}||$.

(b) A sequence (x_n) of positive numbers has the property that for some $a \leq 1$ and some natural number K,

$$\frac{x_{n+1}}{x_n} \geq 1 - \frac{a}{n} \quad \text{for all } n \geq K.$$

Prove that the sequence (nx_{n+1}) is increasing for $n \geq K$. Hence show that there is some positive constant c such that

$$x_{n+1} > \frac{c}{n}$$

for all $n \geq K$.

(c) State the Intermediate Value Theorem.

Suppose the continuous function $f : \mathbb{R} \to \mathbb{R}$ is such that, for all $x \in \mathbb{R}$, there is $y \in \mathbb{R}$ with $f(y) = 2 - f(x)$. Prove that there is some $c \in \mathbb{R}$ such that $f(c) = 1$.

9.(a) A sequence of numbers (x_n) is defined as follows: $x_1 = \alpha$ where $0 < \alpha \leq 1/2$ and, for $n \geq 1$, $x_{n+1} = 2x_n(1 - x_n)$.

Prove that if $x_n \leq 1/2$ then, also, $x_{n+1} \leq 1/2$. [It may help to consider the maximum value of the function $f(x) = 2x(1 - x)$.]

By showing that the sequence (x_n) is increasing and bounded above, deduce that the sequence converges, and determine what its limit is.

(b) What is meant by an 'open' subset of $\mathbb{R}^n$? What is meant by a 'closed' subset of $\mathbb{R}^n$? State a result that characterises closed sets in terms of open sets.

Prove that the subset

$$C = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : y \geq 0 \right\}$$

is a closed subset of $\mathbb{R}^2$.

10.(a) Consider the functions

$$f(x, y, z) = \frac{4}{x} - \frac{2x}{y} + y + 6z - z^3 \qquad (x \neq 0,\ y \neq 0),$$

$$g(x, y, z) = x^2 - xz + y^2 + z^2 - 2x - 3y.$$

For each function, find the critical (or stationary) points and determine the nature of each critical point.

Show that the point $P = (2, 2, 1)$ is on both of the surfaces

$$f(x, y, z) = 7, \quad g(x, y, z) = -3.$$

Find the equations of the tangent planes to each of these surfaces at the point $(2, 2, 1)$.

Show that these two planes are orthogonal.

(b) It is found that the directional derivative of a function $h: \mathbb{R}^3 \rightarrow \mathbb{R}$, at the point (a, b, c) in the direction of $(3, 1, -1)^T$ is equal to $\sqrt{11}$, in the direction of $(1, 1, 0)^T$ it is $\sqrt{2}$, and in the direction of $(2, 1, 2)^T$ it is -2. Find the gradient vector of h at the point (a, b, c). In which direction does the function decrease most rapidly at the point (a, b, c)?

END OF PAPER