1. A man has £7000, and he wants his house to be extended. He considers two possibilities.

(1) Arrange work immediately for £5000 and put the remaining £2000 into a bank account. After one year, in Jan. 2005, he plans to take his money out of the bank, the interest rate being 5%.

(2) Borrow £3000 from a bank and invest the lump sum of £10000, i.e. buy shares. After one year, in Jan. 2005, sell all the shares, pay the loan (along with the annual premium of 5%) and arrange work on the extension.

Suppose $r\%$ is the (annual) return for the shares and $\gamma\%$ is the rate of inflation.

(a) Calculate the amount of money Z (in terms of r and γ) the man has in Jan. 2005 after he implements plan (1) and plan (2). [7 marks]

(b) If $r \sim N(7, 225)$ is normal with mean 7 and variance 225, and $\gamma \sim N(3, 100)$, assuming that r and γ are independent, find the optimal decision using the expected profit criterion $E[Z] \to \max$. [3 marks]

(c) For the optimal decision, find the probability that the man will have in hand less than £2000 in Jan. 2005. [4 marks]

(d) For the same data and $K = 0.02\%$, find the optimal decision using the expected value - variance criterion $E[Z] - K\,Var[Z] \to \max$. [6 marks]

2. A machine is repaired as soon as it breaks down. At the end of T days, preventive maintenance is performed, and the machine becomes as good as new. The number of failures in T days following preventive maintenance is Poisson distributed with parameter $\mu(T) = \int_0^T \Lambda(u)du$. Assume C_1 is the cost of repairing the broken machine, C_2 is the preventive maintenance cost, and $\Lambda(u) = \lambda\sqrt{u}$. (After each preventive maintenance, the machine deteriorates over time until the next preventive maintenance.)

(a) Calculate the expected total cost per day as a function of T, in terms of C_1, C_2, and λ. [9 marks]

(b) Determine the optimal value of T in terms of C_1, C_2 and λ using the expected value criterion. [5 marks]

(c) Calculate the variance of the total cost per day, for the optimal value T^* found in (b). [4 marks]

(d) Take $C_1 = 3$, $C_2 = 8$, $\lambda = 1$ and calculate the optimal value of T. [2 marks]

3. An investor has £5000 capital. He can leave the money in a bank (with a negligible profit), or buy stocks. He estimates the chances that stock appreciates as 60%; in this case he gains £100. Otherwise, he loses £100.

Alternatively, the investor can consult a broker. The broker is not 100 % reliable. If the stock is in fact going to appreciate, then with probability 0.8 his advice will be to invest, if the stock is going to depreciate, then with probability 0.3 his advice will be to invest. If the investor consults the broker he has to pay a fee of £C.

(a) Draw the decision tree for the investor. [4 marks]

(b) Calculate all the probabilities for the chance nodes. [6 marks]

(c) Elaborate the optimal policy for the investor (in terms of C). [6 marks]

(d) What is the maximal value of C that the investor should agree to pay to the broker? [1 marks]

(e) Suppose $C = £10$ and the investor follows the optimal policy. What is the probability that he leaves his money in the bank? [3 marks]

4. Shares of a particular company are trading on 1 Jan. 2004 at £10.00. Assume that each month, the price moves either 20% up or 20% down. Consider a European call option expiring on 1 March 2004, the exercise price being £11.00. Assume that the annual risk free rate is 6%.

(a) Draw the binomial tree of possible price movements. [2 marks]

(b) Suppose on 1 Feb. 2004 the shares price moves up. Calculate the option price on this date. [7 marks]

(c) Calculate the initial option price on 1 Jan. 2004. [8 marks]

(d) Following the optimal ('hedging') strategy, the investor sells 100 options and buys the optimal number of shares, $100 \cdot \Delta$ on 1 Jan. 2004. Suppose shares go up on 1 Feb. What must the investor do on that day? In your explanation, use the particular values of deltas calculated earlier. [3 marks]

5. Suppose that a company had the following values of profit (in £thousands)

2002				2003			
Winter	Spring	Summer	Autumn	Winter	Spring	Summer	Autumn
13.2	14.2	16.0	17.1	17.5	18.4	19.4	20.2

(a) Encode Winter-2002 as 1, Spring-2002 as 2 and so on, and construct the linear regression model

$$p_t = A + Bt, \qquad t = 1, 2, 3, 4$$

for profits in 2002 (four observations only) using the method of least squares. To put it differently, calculate A and B. [7 marks]

(b) Take $S_0 = A + B \cdot 4$ as the initial level and $B_0 = B$ as the initial slope. Fix smoothing constants $\alpha = 0.2$ (level related) and $\beta = 0.3$ (slope related), and perform the exponential smoothing forecasting procedure for year 2003, i.e. calculate $S_1, B_1, \ldots, S_4, B_4$. [8 marks]

(c) Hence, evaluate the forecast for profit in Winter 2004 and in Spring 2004. [2 marks]

(d) What are the values of α and β for the 'last-value forecasting procedure' for this linear trend model? What is the forecast for Winter 2004 and for Spring 2004 in this case? [3 marks]

6. There are two barbers in a barber shop: one for children under 16 and one for adults. The inter-arrival times of children and adults are mutually independent and identically exponentially distributed with mean T_A min. The service times are again independent and identically exponentially distributed with mean T_S min. Suppose there is no space for waiting; a new customer who finds his barber busy goes away unserved.

(a) Describe all possible states of this queueing system and draw the transition diagram. [2 marks]

(b) Write down equations for the steady-state probabilities of the states. [5 marks]

(c) Solve the equations for $T_A = T_S = 15$ min. [3 marks]

(d) What is the steady-state probability to miss a new arriving adult? [3 marks]

(e) If the profit associated with every adult (on average) is $P = £3$, how much money on average gets lost in 1 hour due to rejected adults? [3 marks]

(f) Find all the steady-state probabilities (see Item (c)) for arbitrary positive values of T_A and T_S. [4 marks]

7. (a) For the exponential density function

$$f(t) = \lambda e^{-\lambda t}, \qquad t \geq 0. \qquad \lambda > 0,$$

derive the cumulative distribution function $F(t)$. [3 marks]

(b) Show that an exponential $exp(\lambda)$ random variable T can be calculated by the formula

$$T = G_\lambda(u) = \frac{-\ln u}{\lambda}, \qquad (*)$$

where $u \sim U(0,1)$ is standard uniform. [3 marks]

(c) Having the following 6 realizations of the uniform RV u_i

$$0.65 \qquad 0.23 \qquad 0.68 \qquad 0.10 \qquad 0.77 \qquad 0.82,$$

calculate corresponding values T_i^A using (*) with $\lambda = \lambda_A = 2$. [2 marks]

(d) Having the following 6 realizations of the uniform RV v_i

$$0.58 \qquad 0.14 \qquad 0.10 \qquad 0.85 \qquad 0.11 \qquad 0.85,$$

calculate corresponding values T_i^S using (*) with $\lambda = \lambda_S = 3$. [2 marks]

(e) Let T_i^A be the simulated interarrival times in an M/M/1/∞ queueing system, and T_i^S be the service times. Let $n(t)$ be the number of requests in the system, which is empty initially. Draw the simulated graph of $n(t)$ based on T_i^A and T_i^S calculated earlier. [3 marks]

(f) Calculate the simulated fraction of time on the interval [0,3] when the system is free. Compare it with the analytical steady-state probability $p_0 = \frac{\lambda_S - \lambda_A}{\lambda_S}$ [3 marks]

(g) Calculate the simulated fraction of time on the interval [0,3] when there is exactly one request in the system. Compare it with the analytical steady-state probability $p_1 = p_0 \frac{\lambda_A}{\lambda_S}$. [4 marks]