PAPER CODE NO.
MATH 766

THE UNIVERSITY
of LIVERPOOL

MAY 2004 EXAMINATIONS

Bachelor of Science : Year 3

NUMERICAL ANALYSIS AND SOLUTION OF LINEAR EQUATIONS

TIME ALLOWED : Two Hours and a Half

INSTRUCTIONS TO CANDIDATES

Full marks will be awarded for complete answers to FIVE questions. Only the best 5 answers will be taken into account. Note that each question carries a total of 20 marks that are distributed as stated.

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1. [20 marks]

a) Show that the following real-valued function [7 marks]

$$f(x) = \ln\left(\frac{1-x}{1+x}\right) + 2$$

has the domain $(-1, 1)$ and further by studying its gradient in the domain prove that $f(x)$ has only one root. Verify that this root is $x = (1 - e^{-2})/(1 + e^{-2})$.

Use 2 steps of the Newton-Raphson method to find an approximate solution to $f(x) = 0$, starting from $x^{(0)} = 0.5$. Compute the error in the approximation. [5 marks]

Apply 2 steps of the Secant method to find an approximate solution to $f(x) = 0$, starting from $x^{(0)} = 0.5$ and $x^{(1)} = 0.6$. [5 marks]
(Keep at least 4 decimal digits in calculations.)

b) Set up the general formula for using the Newton-Raphson method for the following three simultaneous nonlinear equations in x_1, x_2, x_3

$$\begin{cases} \sin(x_1 + x_2) - x_1 x_2 = 0, \\ x_1^2 + x_2^2 - 1 = 0, \\ 5x_1^2 + 21x_2^2 - 9x_3 = 0, \end{cases}$$

with some initial guess $\mathbf{x}^{(0)}$. (Do not invert any matrices.) [3 marks]

2.

[20 marks]

For the following linear system

$$\begin{pmatrix} 2 & 4 & 2 \\ 1 & 0 & 3 \\ 3 & 1 & 2 \end{pmatrix} \mathrm{x} = \begin{pmatrix} -4 \\ 4 \\ 3 \end{pmatrix},$$

using exact arithmetic,

i) solve it by Gaussian elimination; [4 marks]

ii) explain how the multipliers and coefficients would give rise to the LU decomposition; [1 mark]

iii) using i) and ii), verify that the $A = LDM$ decomposition with [4 marks]

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 1.5 & 2.5 & 1 \end{pmatrix}, D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -6 \end{pmatrix} \text{ and } M = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix},$$

holds;

iv) compute $\|A\|_1$ and $\|b\|_\infty$; [4 marks]

v) compute L^{-1}, M^{-1} and the 1-norm condition number: $\kappa_1(M) = \text{cond}_1(M)$. [5 marks]

vi) In the first stage of a Gaussian elimination method with partial pivoting for the above system, what permutation P_1 is appropriate? [2 marks]

3. [20 marks]

Given that $A\mathbf{x} = \mathbf{b}$,

$$A = \begin{pmatrix} 9 & 2 & 0 \\ 2 & 8 & -3 \\ 0 & -3 & 7 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 13 \\ 15 \\ 1 \end{pmatrix},$$

i) write out the three equations for the three components of the vector $\mathbf{x}^{n+1}$ for the Jacobi Iteration Method and carry out 2 iterations starting from $\mathbf{x}^0 = \mathbf{0}$. Find the iteration matrix T_J and vector $\mathbf{c}_J$ such that

$$\mathbf{x}^{n+1} = T_J\mathbf{x}^n + \mathbf{c}_J.$$

[6 marks]

ii) write out the three equations for $\mathbf{x}^{n+1}$ for the Gauss-Seidel (GS) Method and carry out 2 iterations starting from $\mathbf{x}^0 = \mathbf{0}$. Write down L, D and U, the lower triangular, diagonal and upper triangular parts of A. Find $(L+D)^{-1}$ and hence the iteration matrix T_{GS} and vector $\mathbf{c}_{GS}$ such that

$$\mathbf{x}^{n+1} = T_{GS}\mathbf{x}^n + \mathbf{c}_{GS}.$$

[8 marks]

iii) use the Gerschgorin theorem to determine if each of the Jacobi and Gauss-Seidel Iteration Methods should converge.

[6 marks]

(Keep at least 4 decimal digits in calculations.)

4.

[20 marks]

For the following matrix A

$$\begin{bmatrix} 9 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & -14 & 8 \end{bmatrix},$$

i) to compute $\lambda(A)$ using the shifted inverse power method, which of the following three possible shifts γ, [5 marks]

$$-50, \quad 1.2, \quad 50,$$

is the most sensible choice (i.e. nearest to one of the eigenvalues of A)? Explain your answer.

ii) given the LU factorisation for $(A - 6I)$ as [15 marks]

$$\begin{bmatrix} 3 & 0 & 1 \\ 0 & -5 & 0 \\ 1 & -14 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/3 & 14/5 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 \\ 0 & -5 & 0 \\ 0 & 0 & 5/3 \end{bmatrix},$$

use the shifted inverse power method for **two** steps to estimate the eigenvalue near $\gamma = 6$ and its eigenvector. Start the iterations from $\mathbf{x}^{(0)} = [-1 \ \ 0 \ \ 1]^T$ and keep at least 4 decimal digits in calculations.

5.

[20 marks]

Solve the following boundary value problem, using the usual finite difference method with $(N+1) \times (M+1) = 3 \times 3$ boxes i.e. 4 interior and uniformly distributed mesh points:

$$-\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = x - y^2, \qquad p = (x, y) \in \Omega$$

where the domain is the square $\Omega = [0, 0.3] \times [0, 0.3] \in R^2$, with the Dirichlet boundary condition $u|_\Gamma = x + y$ given. (Set up the linear system only *without* having to solve it and keep at least 4 decimal digits in calculations.)

6. [20 marks]

a) Compute the Lagrange interpolating polynomial $y = P_3(x)$ of degree 3 that passes through these 4 points (x_j, y_j):

$$(0, \sqrt{17}), \quad (2, \sqrt{5}), \quad (4, 1), \quad (6, \sqrt{5}).$$

[5 marks]

b) The three point quadrature formula can be written as

$$\int_{-1}^{1} f(x)dx = w_0 f(x_0) + w_1 f(x_1) + w_2 f(x_2),$$

where $x_0 = -\sqrt{15}/5, \quad x_1 = 0, \quad x_2 = \sqrt{15}/5$. Find the suitable weights w_j (in exact arithmetic) so that the rule becomes a Gauss type i.e. the rule is exact for degree $0, 1, 2$ polynomials. [6 marks]

Adapt the above rule designed for $[-1, 1]$ to approximately integrate

$$f(x) = \frac{10x^4}{\sqrt{100x^5 + 1}}$$

in $[0, 1]$. Verify that the true integral is $I = (\sqrt{101} - 1)/25$ and compute the absolute error of the approximation to this true value. [5 marks]
(Keep at least 4 decimal digits in calculations.)

Finally verify that the above quadrature rule is also exact for higher order monomials $f(x) = x^3,\ x^4,\ x^5$ and but *not* for $f(x) = x^6$. [4 marks]

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7.

[20 marks]

a) Use both the explicit and implicit Euler methods for solving the initial value problem

$$\frac{dy}{dx} = e^{-x}y - 2y + x, \qquad y(0) = 1$$

to obtain $y(0.5)$ with the step length $h = 0.25$. [10 marks]

b) Combine the nonlinear Newton-Raphson method with the implicit Euler method to solve

$$\frac{dy}{dx} = e^{-x}y - 2y^2 + x, \qquad y(0) = 1$$

for $y(0.5)$ with the step length $h = 0.25$. (Use no more than 3 iterations in each Newton-Raphson step). [10 marks]

(Keep at least 4 decimal digits in calculations.)