Math 766 (May 2004) Solutions

Q1 HW (1) For $f(x) = \ln\left(\frac{1-x}{1+x}\right) + 2$

let $\frac{1-x}{1+x} > 0$: either $\begin{cases} 1-x > 0 \\ 1+x > 0 \end{cases}$, $-1 < x < 1$ m2

or $\begin{cases} 1-x < 0 \\ 1+x < 0 \end{cases}$, no soln.

$\therefore (-1, 1)$ is the domain of $f(x)$.

Within $(-1,1)$, $f(x) = \ln(1-x) - \ln(1+x) + 2$

so $f'(x) = -\frac{1}{1-x} - \frac{1}{1+x} = \frac{-2}{1-x^2} < 0$ m3

as $\underbrace{1-x^2>0}$ $\to$ $f(x)$ monotone & $\to$ unique root in $(-1,1)$.

Let $f(x) = 0$. m2

Then $\frac{1-x}{1+x} = e^{-2} \to x = \frac{1-e^{-2}}{1+e^{-2}} = 0.7616$

To use the N-R, we use $x^{(0)} = 0.5$ and

$$x^{(k+1)} = x^{(k)} - \frac{f(x^{(k)})}{f'(x^{(k)})} = x^{(k)} + \tfrac{1}{2}(1-x^{(k)2})\left[\ln\frac{1-x^{(k)}}{1+x^{(k)}} + 2\right]$$ m2

giving $x^{(1)} = 0.83802$ m2

$x^{(2)} = 0.77416$ $\leftarrow$

$x^{(3)} = 0.76189$ m2

Error $= |0.7616 - 0.77416| = 0.0125$ m1

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Q1 (1) Contd The Secant method reads

$$x^{(k+2)} = x^{(k+1)} - f(x^{(k+1)}) \Big/ \frac{f(x^{(k+1)}) - f(x^{(k)})}{x^{(k+1)} - x^{(k)}}$$

or

$$x^{(k+2)} = x^{(k+1)} - \left[\ln\frac{1-x^{(k+1)}}{1+x^{(k+1)}} + 2\right](x^{(k+1)} - x^{(k)}) \Big/ \ln\left[\frac{1-x^{(k+1)}}{1+x^{(k+1)}} \frac{1+x^{(k)}}{1-x^{(k)}}\right]$$ m2

With $x^{(0)} = 0.5$ and $x^{(1)} = 0.6$, we obtain

Step 1 $x^{(2)} = 0.8133$

$x^{(3)} = 0.7475$ m3

(2) For $F = \begin{pmatrix} f_1(x_1, x_2, x_3) \\ f_2 \\ f_3 \end{pmatrix}$, the N-R method takes the form

$$x^{(k+1)} = x^{(k)} - J^{-1} F(x^{(k)}), \quad k = 0, 1, 2, \dots$$

Here

$$J = \begin{bmatrix} \cos(x_1+x_2) - x_2 & \cos(x_1+x_2) - x_1 & 0 \\ 2x_1 & 2x_2 & 0 \\ 10x_1 & 42x_2 & -9 \end{bmatrix}$$

$$F = \begin{bmatrix} \sin(x_1+x_2) - x_1x_2 \\ x_1^2 + x_2^2 - 1 \\ 5x_1^2 + 21x_2^2 - 9x_3 \end{bmatrix}$$ m3

Q2 HW (1) Use multipliers (no sign) $m_{21} = \frac{1}{2}$, $m_{31} = \frac{3}{2}$ for

Stage 1 $\begin{pmatrix} 2 & 4 & 2 \\ 0 & -2 & 2 \\ 0 & -5 & -1 \end{pmatrix} x = \begin{pmatrix} -4 \\ 6 \\ 9 \end{pmatrix}$ m2

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Use multipliers $m_{32}=\frac{5}{2}$ (no sign) for

Stage 2
$$\begin{pmatrix} 2 & 4 & 2 \\ 0 & -2 & 2 \\ 0 & 0 & -6 \end{pmatrix} x = \begin{pmatrix} -4 \\ 6 \\ -6 \end{pmatrix}$$ m2

$\therefore x_3 = 1,\quad x_2 = -2,\quad x_1 = 1 \rightarrow x = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

(2) In $A=LU$ decomposition, L consists of multipliers used in GE while U is simply the reduced matrix at the last stage # m1

(3) Negate all multipliers to give

$$L = \begin{pmatrix} 1 & & \\ m_{21} & 1 & \\ m_{31} & m_{32} & 1 \end{pmatrix} = \begin{pmatrix} 1 & & \\ 1/2 & 1 & \\ 3/2 & 5/2 & 1 \end{pmatrix}$$

and from (1)

$$A^{(2)} = U = \begin{pmatrix} 2 & 4 & 2 \\ & -2 & 2 \\ & & -6 \end{pmatrix}$$ m4

Take the diagonal matrix D from U i.e.

let $U = DM = \begin{pmatrix} 2 & & \\ & -2 & \\ & & -6 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ & 1 & -1 \\ & & 1 \end{pmatrix}$

$\therefore A = LDM$ is verified. #

(4) $\|A\|_1 = \text{Col}\cdot\max = \max(6, 5, 7) = 7$

$\|b\|_\infty = \max(4, 4, 3) = 4$ m4

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(5) Write $L = \begin{pmatrix} 1 & & \\ 1/2 & 1 & \\ 3/2 & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & 5/2 & 1 \end{pmatrix}$ m2

$$M = \begin{pmatrix} 1 & 2 & 1 \\ & 1 & -1 \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & 1 \\ & 1 & -1 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$\therefore L^{-1} = \begin{pmatrix} 1 & & \\ & 1 & \\ & -\frac{5}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ -\frac{1}{2} & 1 & \\ -\frac{3}{2} & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & \\ -\frac{1}{2} & 1 & \\ -\frac{1}{4} & -\frac{5}{2} & 1 \end{pmatrix}$$ m2

$$M^{-1} = \begin{pmatrix} 1 & -2 & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & -1 \\ & 1 & 1 \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2 & -3 \\ & 1 & 1 \\ & & 1 \end{pmatrix}$$

So $\kappa_1(M) = \|M\|_1 \|M^{-1}\|_1 = 3 \cdot 5 = 15$ m1

(6) Swap rows 1 & 3 $\rightarrow P_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ m2

Q3
HW

(1)
$$\begin{cases} 9x_1^{n+1} = 13 - 2x_2^n \\ 8x_2^{n+1} = 15 - 2x_1^n + 3x_3^n \\ 7x_3^{n+1} = 1 + 3x_2^n \end{cases}$$

or
$$\begin{cases} x_1^{n+1} = \frac{13}{9} - \frac{2}{9}x_2^n \\ x_2^{n+1} = \frac{15}{8} - \frac{1}{4}x_1^n + \frac{3}{8}x_3^n \\ x_3^{n+1} = \frac{1}{7} + \frac{3}{7}x_2^n \end{cases}$$ m4

i.e. $x^{n+1} = T_J x^n + c_J$ with

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$$T_J = \begin{pmatrix} 0 & \frac{-2}{9} & 0 \\ -\frac{1}{4} & 0 & \frac{3}{8} \\ 0 & \frac{3}{7} & 0 \end{pmatrix}, \quad C_J = \begin{pmatrix} \frac{13}{9} \\ \frac{15}{8} \\ \frac{1}{7} \end{pmatrix} \begin{matrix} 1.4444 \\ 1.875 \\ 0.1429 \end{matrix}$$ m2

Here $L = \begin{pmatrix} 0 & & \\ 2 & 0 & \\ 0 & -3 & 0 \end{pmatrix}$, $D = \begin{pmatrix} 9 & & \\ & 8 & \\ & & 7 \end{pmatrix}$, $U = \begin{pmatrix} 0 & 2 & 0 \\ & 0 & -3 \\ & & 0 \end{pmatrix}$ m2

$\therefore$ $x^{(1)} = C_J$ and $x^{(2)} = [1.0278, 1.5675, 0.9464]^T$.

(2) For GS,

$$\begin{cases} 9x_1^{n+1} = 13 - 2x_2^n \\ 8x_2^{n+1} = 15 - 2x_1^{n+1} + 3x_3^n \\ 7x_3^{n+1} = 1 + 3x_2^{n+1} \end{cases}$$

or

$$\begin{cases} 9x_1^{n+1} = 13 - 2x_2^n, \\ 2x_1^{n+1} + 8x_2^{n+1} = 15 + 3x_3^n, \\ -3x_2^{n+1} + 7x_3^{n+1} = 1, \end{cases}$$ m3

i.e. $$\begin{pmatrix} 9 & & \\ 2 & 8 & \\ 0 & -3 & 7 \end{pmatrix} x^{n+1} = \begin{pmatrix} 13 \\ 15 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 & 2 & \\ & 0 & -3 \\ & & 0 \end{pmatrix} x^n$$

or $$x^{n+1} = (D+L)^{-1}b - (D+L)^{-1}Ux^n$$
$$= T_{GS}x^n + C_{GS}$$

where $$(D+L)^{-1} = (I + D^{-1}L)^{-1}D^{-1}$$
$$= \begin{pmatrix} 1 & & \\ \frac{1}{4} & 1 & \\ 0 & -\frac{3}{7} & 1 \end{pmatrix}^{-1} \begin{pmatrix} \frac{1}{9} & & \\ & \frac{1}{8} & \\ & & \frac{1}{7} \end{pmatrix}$$

$$(D+L)^{-1} = \begin{pmatrix} 1 & & \\ & 1 & \\ & \frac{3}{7} & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ -\frac{1}{4} & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{9} & & \\ & \frac{1}{8} & \\ & & \frac{1}{7} \end{pmatrix}$$
$$= \begin{pmatrix} 1 & & \\ -\frac{1}{4} & 1 & \\ -\frac{3}{28} & \frac{3}{7} & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{9} & & \\ & \frac{1}{8} & \\ & & \frac{1}{7} \end{pmatrix}$$ m2
$$= \begin{pmatrix} \frac{1}{9} & & \\ -\frac{1}{36} & 1/8 & \\ -\frac{1}{84} & \frac{3}{56} & \frac{1}{7} \end{pmatrix} = \begin{pmatrix} 0.1111 & 0 & 0 \\ -0.0278 & 0.125 & 0 \\ -0.0119 & 0.0536 & 0.1429 \end{pmatrix},$$

$$\therefore T_{GS} = -(D+L)^{-1}U = -\begin{pmatrix} \frac{1}{9} & & \\ -\frac{1}{36} & \frac{1}{8} & \\ -\frac{1}{84} & \frac{3}{56} & \frac{1}{7} \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 \\ & 0 & -3 \\ & & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -2/9 & 0 \\ 0 & 1/18 & 3/8 \\ 0 & 1/42 & 9/56 \end{pmatrix} = \begin{pmatrix} 0 & -0.222 & 0 \\ 0 & 0.0556 & 0.375 \\ 0 & 0.0238 & 0.1607 \end{pmatrix}$$

$$C_{GS} = (D+L)^{-1}b$$
$$= \begin{pmatrix} 13/9 \\ 109/72 \\ 19/24 \end{pmatrix} = \begin{pmatrix} 1.4444 \\ 1.5139 \\ 0.7917 \end{pmatrix}$$ m1

$\therefore$ $x^{(1)} = C_{GS}$

$x^{(2)} = [1.1080, 1.8949, 0.9549]^T$.

(3) As from (1), $T_J = \begin{pmatrix} 0 & -0.2222 & 0 \\ -0.25 & 0 & 0.375 \\ 0 & 0.4286 & 0 \end{pmatrix}$

The Gerschgorin theorem gives

$$|\lambda(T_J)| \le 0.25 + 0.375 = 0.625 < 1$$

and for T_{GS} (concentrate on the largest eig)

$$|\lambda(T_{GS})| \le 0.0556 + 0.375 = 0.4306 < 1$$

$\therefore$ $\rho(T_J) < 1$, $\rho(T_{GS}) < 1$

Both methods should converge!!

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Q4 (BW)

(1) To estimate where the eigenvalues are, use again the Gerschgorin theorem:

$$\begin{cases} |\lambda-\cancel{9}| \le 1 \\ \lambda = 1 \\ |\lambda-8| \le 15 \end{cases}$$

M5

$\therefore \gamma = 1.2$ is the closest to $\lambda_2 = 1$.

(2) The Power method will be applied to matrix $B=(A-6I)^{-1}=(A-\gamma I)^{-1}$

$$=(LU)^{-1}=U^{-1}L^{-1}$$

$$=\begin{pmatrix} 3 & 0 & 1 \\ & -5 & 0 \\ & & 5/3 \end{pmatrix}^{-1}\begin{pmatrix} 1 & & \\ & 1 & \\ 1/3 & 14/5 & 1 \end{pmatrix}^{-1}$$

$$=\begin{pmatrix} 1 & 0 & \frac{1}{3} \\ & 1 & 0 \\ & & 1 \end{pmatrix}^{-1}\begin{pmatrix} 3 & & \\ & -5 & \\ & & \frac{5}{3} \end{pmatrix}^{-1}\begin{pmatrix} 1 & & \\ & 1 & \\ & -\frac{14}{5} & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ 0 & 1 & \\ -\frac{1}{3} & & 1 \end{pmatrix}$$

$$=\underbrace{\begin{pmatrix} 1 & 0 & -\frac{1}{3} \\ & 1 & 0 \\ & & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{3} & & \\ & -\frac{1}{5} & \\ & & \frac{3}{5} \end{pmatrix}\begin{pmatrix} 1 & & \\ 0 & 1 & \\ -\frac{1}{3} & -\frac{14}{5} & 1 \end{pmatrix}}_{\text{COULD LEAVE LIKE THIS !!!}}$$

or $=\begin{pmatrix} 0.4 & 0.56 & -0.2 \\ 0 & -0.2 & 0 \\ -0.2 & -1.68 & 0.6 \end{pmatrix}$ if multiplied out M5

* The method for B will be $x^{(0)}=\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$$k=0,1,2 \quad \begin{cases} y^{(k+1)} = Bx^{(k)} \\ \mu = y_m \quad \text{if } |y_m| = \max_j |y_j^{(k+1)}| \\ x^{(k+1)} = y^{(k+1)}/\mu \end{cases}$$

M1

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This yields the sequences

$$\begin{cases} y^{(1)} = \begin{pmatrix} -0.6 \\ 0 \\ 0.8 \end{pmatrix}, \quad \mu = 0.8 \quad (m=3) \\ x^{(1)} = y^{(1)}/\mu = \begin{pmatrix} -0.75 \\ 0 \\ 1 \end{pmatrix} \end{cases}$$

M4

$$\begin{cases} y^{(2)} = \begin{pmatrix} -0.5 \\ 0 \\ 0.75 \end{pmatrix}, \quad \mu = 0.75 \quad (m=3) \\ x^{(2)} = y^{(2)}/\mu = \begin{pmatrix} -0.6667 \\ 0 \\ 1 \end{pmatrix} \end{cases}$$

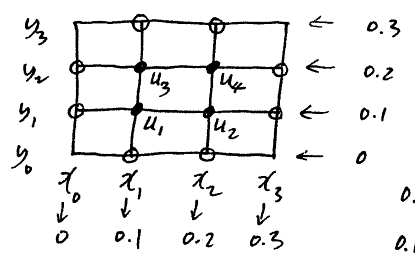
M4

$\therefore \lambda(B) = \mu = 0.75$ from 2 steps

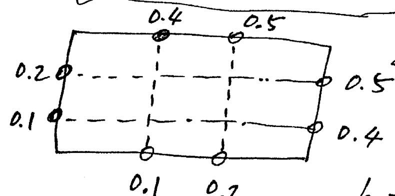
$x = x^{(2)}$ M1

so $\lambda(A) = \gamma + \frac{1}{\mu} = 6 + \frac{1}{0.75} = 7.3333$

Q5 (HW)



$$\boxed{\begin{matrix} u_1 = u_{11}, & u_2 = u_{21} \\ u_3 = u_{12}, & u_4 = u_{22} \end{matrix}}$$



BC's

$h = h_x = h_y = 0.1$

At (x_1, y_1) for u_1

$$\frac{2u_1-u_2-0.1}{h^2}+\frac{2u_1-u_3-0.1}{h^2}=0.1-0.1^2$$

1st eqn: $4u_1-u_2-u_3 = 0.2+0.01\times 0.09 = 0.2009$

At $(x_2, y_1) = (0.2, 0.1)$ for u_2

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$$\frac{2u_2-u_1-0.4}{h^2}+\frac{2u_2-u_4-0.2}{h^2}=0.2-0.1^2,$$

2nd eqn: $4u_2-u_1-u_4=0.6+0.01\times 0.19=0.6019$

At $(x_1, y_2)=(0.1, 0.2)$ for u_3

$$\frac{2u_3-u_4-0.2}{h^2}+\frac{2u_3-u_1-0.4}{h^2}=0.1-0.2^2,$$

3rd eqn. $4u_3-u_1-u_4=0.6+0.01\times 0.06=0.6006$

At $(x_2, y_2)=(0.2, 0.2)$ for u_4

$$\frac{2u_4-u_3-0.5}{h^2}+\frac{2u_4-u_2-0.5}{h^2}=0.2-0.2^2,$$

4th eqn: $4u_4-u_2-u_3=1+0.01\times 0.16=1.0016$

$$\therefore \left[\begin{array}{cc|cc} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ \hline -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{array}\right]\begin{pmatrix}u_1\\u_2\\u_3\\u_4\end{pmatrix}=\begin{pmatrix}0.2009\\0.6019\\0.6006\\1.0016\end{pmatrix}$$

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Q6 (1) Lagrange Polynomial $P(x)$ given by

SC, HW

$$P(x)=\frac{(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)}y_1+\frac{(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)}y_2+$$

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$$\frac{(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_1)(x_3-x_2)(x_3-x_4)}y_3+\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)}y_4$$

$$=\frac{(x-2)(x-4)(x-6)}{(-2)(-4)(-6)}\sqrt{17}+\frac{x(x-4)(x-6)}{2(2-4)(2-6)}\sqrt{5}+\frac{x(x-2)(x-6)}{4(4-2)(4-6)}1+\frac{x(x-2)(x-4)}{6(6-2)(6-4)}\sqrt{5}$$

M5

$$=-\frac{\sqrt{17}}{48}(x-2)(x-4)(x-6)+\frac{\sqrt{5}}{16}x(x-4)(x-6)-\frac{1}{16}x(x-2)(x-6)+\frac{\sqrt{5}}{48}x(x-2)(x-4)$$

(2) $\because \int_{-1}^{1}f(x)dx=W_0f(-\frac{\sqrt{15}}{5})+W_1f(0)+W_2f(\frac{\sqrt{15}}{5}),$

$\therefore$ Take $f=x^0$

$$\int_{-1}^{1}dx=W_0+W_1+W_2=2$$

Take $f=x^1$

$$\int_{-1}^{1}x\,dx=-\frac{\sqrt{15}}{5}W_0+\frac{\sqrt{15}}{5}W_2=0$$

$\therefore W_2=W_0$

Take $f=x^2$

M6

$$\int_{-1}^{1}x^2dx=\frac{3}{5}W_0+\frac{3}{5}W_2=\frac{2}{3}$$

$$\therefore \begin{cases} W_0+W_1+W_2=2, \\ W_0=W_2, \\ W_0+W_2=\frac{10}{9}, \end{cases} \rightarrow \begin{cases} W_0=W_2=\frac{5}{9} \\ W_1=\frac{11}{9} \end{cases}$$

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For the $[0,1]$ case, let $y = 2x-1$, $x = \frac{y+1}{2}$:

$$\int_0^1 f(x)\,dx = \frac{1}{2}\int_{-1}^{1} f\left(\frac{y+1}{2}\right)dy, \quad f(x) = \frac{10x^4}{\sqrt{100x^5+1}}$$

$$= \frac{1}{2}\left(W_0 f(z_0) + W_1 f(z_1) + W_2 f(z_2)\right)$$

$$= \frac{1}{2}\left\{\frac{5}{9} f(z_0) + \frac{8}{9} f(z_1) + \frac{5}{9} f(z_2)\right)$$

with $z_0 = \frac{x_0+1}{2} = 0.1127$, $f(z_0) = 0.0016$,

$z_1 = \frac{x_1+1}{2} = 0.5000$, $f(z_1) = 0.3077$,

$z_2 = \frac{x_2+1}{2} = 0.8873$, $f(z_2) = 0.8283$ M3

leading to the approximation

$$A = \frac{1}{2}\left[\frac{5}{9}(0.0016 + 0.8283) + \frac{8}{9} \times 0.3077\right]$$

$$= \underline{0.3673}$$

While the exact integral (using the transform $y = x^5 + \frac{1}{100}$)

$$I = \int_0^1 \frac{10x^4}{\sqrt{100x^5+1}}dx = \int_0^1 \frac{x^4\,dx}{\sqrt{x^5+\frac{1}{100}}} = \int_{\frac{1}{100}}^{1+\frac{1}{100}} \frac{dy}{\sqrt{y}}$$

$$= \frac{2}{5}\sqrt{y}\Big|_{1/100}^{1+\frac{1}{100}} = \frac{1}{25}\left(\sqrt{101}-1\right) \doteq 0.3620$$

$\therefore$ Abs error $= \underline{0.0053}$. m2 m1

Finally for x^3: $\begin{cases} \text{Exact } I = \int_{-1}^{1} x^3 dx = 0 \\ \text{Numer} = \frac{5}{9}(-x_0^3 + x_0^3) = 0 \end{cases}$

$\therefore$ Exact for x^3

Similarly for x^4:

$$\left\| \begin{array}{l} \text{Exact } I = \int_{-1}^{1} x^4 dx = \frac{2}{5} \\ \text{Numer} = \frac{5}{9}(x_0^4 + x_0^4) = \frac{5}{9}\frac{2\times 9}{25} = \frac{2}{5} \end{array} \right.$$

$\therefore$ Yes

for x^5: $\left\| \begin{array}{l} \text{Exact } I = \int_{-1}^{1} x^5 dx = 0 \\ \text{Numer} = \frac{5}{9}(-x_0^5 + x_0^5) = 0 \end{array} \right.$ M4

Yes

for x^6: $\begin{cases} \text{Exact } I = \int_{-1}^{1} x^6 dx = \frac{2}{7} \\ \text{Numer} = \frac{5}{9}(x_0^6 + x_0^6) = \frac{6}{25} = \frac{2}{8.333} < \frac{2}{7} \end{cases}$

$\therefore$ No

Q7 BW

(1) Explicit scheme:

$$\frac{y_{n+1} - y_n}{h} = f(x_n, y_n), \quad n \geq 0$$

With $y_0 = 1$ and $h = 0.25$, $x_0 = 0$,

$$y_1 = y_0 + h\left[e^{-x_0} y_0 - 2y_0 + x_0\right]$$

$$= 1 + h[-1] = \underline{0.75}, \quad x_1 = 0.25,$$

$$y_2 = y_1 + h\left[e^{-x_1} y_1 - 2y_1 + x_1\right]$$

$$= 0.75 + h[-0.6659] = \underline{\underline{0.5835}}$$ M5

Next we use the

Implicit scheme:

$$\frac{y_{n+1}-y_n}{h} = f(x_{n+1}, y_{n+1}), \quad n \geq 0$$

At $n=0$, $y_0=1$ and $x_1=h=0.25$,

$$y_1-y_0 = (e^{-x_1}y_1 - 2y_1 + x_1)h$$

$\therefore [1-h(e^{-x_1}-2)]y_1 = y_0 + hx_1$,

$\therefore y_1 = 1.0625/1.3053 = \underline{0.814}$

At $n=1$, $y_1=0.814$ and $x_2=2h=0.5$, MS

$$y_2-y_1 = (e^{-x_2}y_2 - 2y_2 + x_2)h,$$

$\therefore [1-h(e^{-x_2}-2)]y_2 = y_1 + hx_2$,

$\therefore y_2 = 0.939/1.3484 = \underline{\underline{0.6964}}$. ✱

(2) $\therefore \frac{y_{n+1}-y_n}{h} = f(x_{n+1}, y_{n+1}), \quad n \geq 0$

At $n=0$, $y_0=1$, and $x_1=h=0.25$:

$$y_1-y_0 = (e^{-x_1}y_1 - 2y_1^2 + x_1)h,$$

from solving $f(y) = he^{-x_1}y - 2y^2h + y_0 - y + hx_1 = 0$

for $y=y_1$ via the Newton-Raphson next

$$f'(y) = he^{-x_1} - 4hy - 1$$

Thus, with $y^{(0)}=y_0=1$ initially, we obtain

$$y^{(1)} = y^{(0)} - \frac{f(y^{(0)})}{f'(y^{(0)})} = 0.86551,$$

$$y^{(2)} = y^{(1)} - \frac{f(y^{(1)})}{f'(y^{(1)})} = 0.86009,$$

$$y^{(3)} = y^{(2)} - \frac{f(y^{(2)})}{f'(y^{(2)})} = \underline{0.86009}.$$ MS

$\therefore y_1 = 0.86009$ at $x_1=0.25$.

Finally at $n=1$, $y_1=0.86009$, $x_2=2h=0.5$:

$$\frac{y_2-y_1}{h} = e^{-x_2}y_2 - 2y_2^2 + x_2.$$

We now use the NR to solve the eqn

$$f(y) = he^{-x_2}y - 2y^2h + y_1 - y + hx_2 = 0$$

for $y=y_2$ starting from $y^{(0)}=y_1=0.86009$.

So
$$y^{(1)} = y^{(0)} - \frac{f(y^{(0)})}{f'(y^{(0)})} = 0.79309,$$ MS

$$y^{(2)} = y^{(1)} - \frac{f(y^{(1)})}{f'(y^{(1)})} = 0.79172,$$

$$y^{(3)} = y^{(2)} - \frac{f(y^{(2)})}{f'(y^{(2)})} = \underline{\underline{0.79172}},$$

i.e. $\underline{\underline{y_2 = 0.79172 \text{ at } x_2 = 0.5}}$ ✱