1.

(a) [5 marks]

Show that the following real-valued function

$$f(x) = \ln\left(\frac{1+x}{1-x}\right) + 1$$

has the domain $(-1, 1)$ and further by studying its gradient in the domain prove that $f(x)$ has only one root. Verify that this root is $x = (1-e)/(1+e)$.

[6 marks]

Use 2 steps of the Newton-Raphson method to find an approximate solution to $f(x) = 0$, starting from $x^{(0)} = 0$. Compute the error in the approximation.

[5 marks]

Use 2 steps of the Secant method to find an approximate solution to $f(x) = 0$, starting from $x^{(0)} = 0, x^{(1)} = -0.5$.

(Keep at least 4 decimal digits in calculations.)

(b) [4 marks]

Set up the general formula for using the Newton-Raphson method for the following three simultaneous nonlinear equations in x_1, x_2, x_3:

$$\begin{cases} \ln(x_1/2) + x_2 = 0, \\ x_1 x_3 + 1 = 0, \\ x_2^2 + 5x_3^2 - 1 = 0, \end{cases}$$

with some initial guess $\mathbf{x}^{(0)}$. (Do NOT invert any matrices.)

2.

For the following linear system $Ax = b$ with

$$A = \begin{pmatrix} 2 & 0 & 4 \\ 4 & 3 & 5 \\ 0 & -3 & 4 \end{pmatrix}, b = \begin{pmatrix} -2 \\ -1 \\ -4 \end{pmatrix}$$

(i) solve it by Gaussian elimination; [5 marks]

(ii) use the multipliers to form both the LU and the LDM decompositions: $A = LU$ and $A = LDM$; [7 marks]

(iii) compute $\|A\|_1$ and $\|b\|_\infty$; [4 marks]

(iv) find the 1-norm and ∞-norm condition numbers $\kappa_1(A)$ and $\kappa_\infty(A)$, using $\|A^{-1}\|_\infty = 17/2$, $\|A^{-1}\|_1 = 55/6$. [4 marks]

3. Consider the linear system $Ax = b$, where

$$A = \begin{pmatrix} 5 & -1 & 2 \\ 1 & -3 & -1 \\ 1 & 0 & 7 \end{pmatrix}, \quad b = \begin{pmatrix} 5 \\ 11 \\ 7 \end{pmatrix}.$$

Using exact arithmetic (i.e. fractions):

(i) Write down the 3 equations for the 3 components of the vector $x^{(n+1)}$ for the Jacobi iteration method and carry out 2 iterations starting from $x^{(0)} = \mathbf{0}$. Find the iteration matrix T_J and the vector c_J such that

$$x^{(n+1)} = T_J x^{(n)} + c_J.$$

[7 marks]

(ii) Find $(L + D)^{-1}$, where L and D are the lower diagonal and the diagonal parts of A respectively, and hence compute the iteration matrix T_{GS} and the vector c_{GS} of the Gauss-Seidel iteration method such that

$$x^{(n+1)} = T_{GS} x^{(n)} + c_{GS}.$$

[8 marks]

(iii) Use the Gerschgorin theorem to determine if each of the Jacobi and Gauss-Seidel Iteration methods should converge. [5 marks]

4.

Consider the following matrix

$$A = \begin{pmatrix} 3 & 0 & 1 \\ 2 & -1 & -1 \\ 0 & 1 & 7 \end{pmatrix}.$$

(i) Use the Gerschgorin theorem to locate all three eigenvalues. [5 marks]

(ii) Given the LU factorization for $(A - 7I)$ as

$$(A-7I) = \begin{pmatrix} -4 & 0 & 1 \\ 2 & -8 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -1/8 & 1 \end{pmatrix} \begin{pmatrix} -4 & 0 & 1 \\ 0 & -8 & -1/2 \\ 0 & 0 & -1/16 \end{pmatrix}$$

use the shifted inverse power method for two steps to estimate the eigenvalue near $\gamma = 7$ and its eigenvector. Start the iteration from $x^{(0)} = [0, 0, 1]^T$ and keep at least 2 decimal places throughout your calculations.

[15 marks]

5. [20 marks]

Consider the following boundary value problem, by the usual finite difference method with $(N+1) \times (M+1) = 3 \times 3$ boxes, i.e. 4 interior and uniformly distributed mesh points:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2(x+y)^2, \quad p = (x, y) \in \Omega$$

where the domain is the square $\Omega = [0, 0.3] \times [0, 0.3] \in R^2$, with the Dirichlet boundary condition $u = x$ given. Set up the linear system in matrix form, but do not solve it. Keep at least 4 decimal digits in calculations.

6.

(a) Compute the Lagrange interpolating polynomial $y = P_3(x)$ of degree 3 that passes through these 4 points (x_j, y_j)

$$(0,2),\ (\sqrt{2},3),\ (-1,\sqrt{7}),\ (5,\sqrt{13}).$$

[3 marks]

(b) The three point quadrature formula can be written as

$$\int_{-1}^{1} f(x)dx = w_0 f(x_0) + w_1 f(x_1) + w_2 f(x_2)$$

where $x_0 = -\sqrt{15}/5,\ x_1 = 0,\ x_2 = \sqrt{15}/5.$

Find suitable weights w_j (in exact arithmetic) so that the rule becomes a Gauss type, i.e. the rule is exact for degree $0, 1, 2$ polynomials. [5 marks]

Adapt the above rule designed for $[-1, 1]$ to approximate

$$I = \int_0^1 \frac{3x^3 dx}{\sqrt{2 + 10x^4}}.$$

Verify that the true integral is $I = \frac{3\sqrt{3}}{10} - \frac{3\sqrt{2}}{20}$ and compute the absolute error of the approximation to this true value (keeping at least 4 decimal digits in calculations) [9 marks]

Finally verify that the above quadrature rule is also exact for higher order polynomials $f(x) = x^3, x^4, x^5$, but not for $f(x) = x^6$. [3 marks]

7.

(a) Use the explicit Euler method to solve the initial value problem

$$\frac{dy}{dx} = \ln(x + y), \quad y(0) = 2$$

to obtain y(0.2) with the step length $h = 0.1$. (Keep at least 4 decimal digits in calculations)

Explain how your method would change if you were to use the implicit Euler method.

[8 marks]

(b) Using the composite Trapezium rule (with 2 subintervals) in a collocation method, set up the linear system to find the numerical solution of the following integral equation

$$7u(x) - \int_0^1 \cos(x + y)u(y)dy = 2x + 1 \quad x \in [0, 1].$$

Do not solve the system. [12 marks]