THE UNIVERSITY
of LIVERPOOL

1. Keep at least 4 decimal digits in calculations.

(a) Show that the following real-valued function

$$f(x) = \ln\left(\frac{1-x}{1+x}\right) + 1$$

has the domain $(-1, 1)$ and further by studying its derivative in the domain prove that $f(x)$ has exactly one root. Verify that this root is $x = (e-1)/(1+e)$.

[4 marks]

Use 2 steps of the Newton-Raphson method to find an approximate solution to $f(x) = 0$, starting from $x^{(0)} = 0$. Compute the error in the first and second steps, and estimate the error in the third step, given that the Newton-Raphson method exhibits quadratic convergence.

[7 marks]

(b) Carry out 1 step of the Newton-Raphson method starting at the point $x^{(0)} = (1, -1, 2)^T$ for the following three simultaneous nonlinear equations in x_1, x_2, x_3:

$$\begin{cases} 4x_1 - x_3^2 = 0, \\ x_2^2 + 2x_1^2 - 3 = 0, \\ x_2 x_3 - 6\cos(x_2 + x_3) = 0. \end{cases}$$

[9 marks]

2. Given that $Ax = b$, where

$$A = \begin{pmatrix} 4 & 1 & 2 \\ 2 & -5 & 1 \\ 0 & 1 & 6 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 5 \\ 12 \end{pmatrix}.$$

Using exact arithmetic (i.e. fractions):

(i) Write down the 3 equations for the 3 components of the vector $x^{(n+1)}$ for the Jacobi iteration method and carry out 2 iterations starting from $x^{(0)} = (0,0,0)^T$. Find the iteration matrix T_J and the vector c_J such that

$$x^{(n+1)} = T_J x^{(n)} + c_J.$$

[7 marks]

(ii) Compute the iteration matrix T_{GS} and the vector c_{GS} of the Gauss-Seidel iteration method such that

$$x^{(n+1)} = T_{GS} x^{(n)} + c_{GS}.$$

Carry out 1 iteration of this method using this iteration matrix and vector, starting from $x^{(0)} = (0,0,0)^T$.

[9 marks]

(iii) Use the Gerschgorin theorem to determine if each of the Jacobi and Gauss-Seidel iteration methods should converge.

[4 marks]

3. Consider the following matrix A,

$$A = \begin{pmatrix} -5 & 2 & -1 \\ 0 & -1 & 0 \\ -1 & 3 & 7 \end{pmatrix}.$$

(i) Use the Gerschgorin theorem to locate all three eigenvalues.

[3 marks]

(ii) Use the power method for 2 steps to compute the dominant eigenvalue and its corresponding eigenvector. Start from $z = [0, 0, 1]^T$ and keep at least 4 decimal digits in calculations.

[5 marks]

(iii) Use the shifted inverse power method for 2 steps to compute the eigenvalue near $\gamma = -5$ and its corresponding eigenvector. Start from $z = [1, 0, 0]^T$ and keep at least 4 decimal digits in calculations. You can use the result that

$$(A + 5I)^{-1} = \begin{pmatrix} -12 & 6.75 & -1 \\ 0 & 0.25 & 0 \\ -1 & 0.5 & 0 \end{pmatrix}.$$

[12 marks]

4.

(a) Consider the solution of the following boundary value problem, by the usual finite difference method with 3×3 boxes, i.e. 4 interior and uniformly distributed mesh points:

$$(1+2y)\frac{\partial^2 u}{\partial x^2} + (1+x)\frac{\partial^2 u}{\partial y^2} = (x+y+1)^2, \quad p = (x,y) \in \Omega$$

where the domain is the square $\Omega = [0, 0.3] \times [-0.1, 0.2] \in R^2$, with the Dirichlet boundary condition $u = 2y$ given.

Set up the linear system in matrix form, but do not solve it. Keep at least 4 decimal digits in calculations.

[16 marks]

(b) Using the usual finite central difference formulae, write down the finite difference equation for v_{ijk} at a general point (x_i, y_j, z_k) in the following 3D equation:

$$(1+2y)\frac{\partial^2 v}{\partial x^2} + (1+x)\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = (x+y+z+1)^2.$$

[4 marks]

5.

(a) Use the explicit Euler method to solve the initial value problem

$$\frac{dx}{dt} = 3x(1 - 0.1x), \quad x(0) = 1,$$

to obtain $x(0.2)$ with the step length $h = 0.1$ (keep at least 4 decimal places in calculations). Compare your result with the exact solution:

$$x(t) = \frac{10e^{3t}}{10 + (e^{3t} - 1)}.$$

Suggest how you could improve the accuracy of the numerical estimate.

Explain how your method would change if you were to use the implicit Euler method.

[8 marks]

(b) Using the composite Trapezium rule (with 2 subintervals) in a collocation method, set up the linear system to find the numerical solution of the following integral equation

$$2u(x) - \int_0^1 e^{xy-1} u(y)dy = x - 1, \quad x \in [0, 1].$$

Do not solve the system.

[12 marks]

6. Consider the matrix A:

$$A = \begin{pmatrix} 2 & 3 & 0 & 0 \\ 1 & 2 & 0 & -1 \\ 4 & 8 & -1 & 0 \\ 0 & 2 & 1 & -3 \end{pmatrix}.$$

Using exact arithmetic (i.e. fractions), compute the $PA = LU$ decomposition of matrix A with partial pivoting. Use the decomposition to solve the system $Ax = [1, 1, 2, 4]^T$. Explain the computational benefits of pivoting.

[20 marks]

7.

a) Consider the linear system $Ax = b$ where

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 4 & 5 & 4 \\ 2 & 10 & 11 \end{pmatrix}, b = \begin{pmatrix} 4 \\ 12 \\ 15 \end{pmatrix}.$$

Solve it by Gaussian elimination, and find the $A = LDM$ decomposition. Use this LDM decomposition to solve

$$\begin{pmatrix} 2 & 1 & 0 \\ 4 & 5 & 4 \\ 2 & 10 & 11 \end{pmatrix} x = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

Explain the computational benefits of matrix factorization.

[12 marks]

b) Obtain the Crout LU factorization for the following tridiagonal matrix, B:

$$B = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 4 & 5 & 4 & 0 & 0 \\ 0 & 5 & 4 & -1 & 0 \\ 0 & 0 & 4 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

[8 marks]