MATH764 January 2007 Exam Solutions

All questions similar to seen exercises except where marked as Bookwork (B) or Unseen (U).

1. (a) Probability mass function of S_3:

$$\begin{aligned}
P(S_3 = 0) &= P(Y_1 = Y_2 = Y_3 = 0) = \frac{1}{2} \times \frac{3}{4} \times \frac{7}{8} = \frac{21}{64} \\
P(S_3 = 1) &= P(Y_1 = 1, Y_2 = Y_3 = 0) + P(Y_2 = 1, Y_1 = Y_3 = 0) \\
&\quad + P(Y_3 = 1, Y_1 = Y_2 = 0) \\
&= \frac{1}{2} \times \frac{3}{4} \times \frac{7}{8} + \frac{1}{4} \times \frac{1}{2} \times \frac{7}{8} + \frac{1}{8} \times \frac{1}{2} \times \frac{3}{4} = \frac{21+7+3}{64} = \frac{31}{64} \\
P(S_3 = 2) &= P(Y_1 = Y_2 = 1, Y_3 = 0) + P(Y_2 = Y_3 = 1, Y_1 = 0) \\
&\quad + P(Y_3 = Y_1 = 1, Y_2 = 0) \\
&= \frac{1}{2} \times \frac{1}{4} \times \frac{7}{8} + \frac{1}{4} \times \frac{1}{8} \times \frac{1}{2} + \frac{1}{8} \times \frac{1}{2} \times \frac{3}{4} = \frac{7+1+3}{64} = \frac{11}{64} \\
P(S_3 = 3) &= P(Y_1 = Y_2 = Y_3 = 1) = \frac{1}{2} \times \frac{1}{4} \times \frac{1}{8} = \frac{1}{64}
\end{aligned}$$

(b) $P(S_n = n) = P(Y_1 = Y_2 = \cdots = Y_n = 1) = (1/2)^{1+2+\cdots+n} = (1/2)^{n(n+1)/2}$.

(c) CLT can't be used to approximate distribution of S_n, because the summands $Y_1, \ldots, Y_n$ are not identically distributed.

(d) $E[Y_i] = (1/2)^i$ and $\text{Var}[Y_i] = (1/2)^i\left(1 - (1/2)^i\right)$, so

$$\begin{aligned}
E[S_n] &= \sum_{i=1}^{n} E[Y_i] = \sum_{i=1}^{n} \left(\frac{1}{2}\right)^i = \frac{1-(1/2)^{n+1}}{1-(1/2)} - 1 = 2\left(1 - (1/2)^{n+1}\right) - 1 \\
&= 1 - (1/2)^n \\
\text{Var}[S_n] &= \sum_{i=1}^{n} \text{Var}[Y_i] = \sum_{i=1}^{n} \left(\frac{1}{2}\right)^i - \sum_{i=1}^{n} \left(\frac{1}{4}\right)^i = 1 - (1/2)^n - \left(\frac{1-(1/4)^{n+1}}{1-(1/4)} - 1\right) \\
&= 2 - (1/2)^n - (4/3)\left(1 - (1/4)^{n+1}\right) = (2/3) - (1/2)^n + (1/3)(1/4)^n
\end{aligned}$$

(e) $E[\bar{Y}_n] = E[S_n/n] = E[S_n]/n = \left(1 - (1/2)^n\right)/n \to 0$ as $n \to \infty$.
$\text{Var}[\bar{Y}_n] = \text{Var}[S_n/n] = \text{Var}[S_n]/n^2 = \left((2/3) - (1/2)^n + (1/3)(1/4)^n\right)/n^2 \to 0$ as $n \to \infty$.

This seems intuitively reasonable because $P(Y_i = 0) \to 1$ as $n \to \infty$, so expect only finitely many of the Y_i to be non-zero, so that the limiting distribution of Y_n/n will be concentrated at zero with probability 1. (NB: Borel-Cantelli lemmas not covered in this module.)

2. (a) Memoryless property: for $t, s > 0$, $P(T > t+s \mid T > t) = P(T > s)$.

Intuitively, this means that knowledge that an item whose lifetime is distributed as T has already survived for time t does not alter the distribution of the remaining lifetime from t onwards.

For exponential distribution,

$$\begin{aligned} P(T > t+s \mid T > t) &= \frac{P(T > t+s \text{ and } T > t)}{P(T > t)} = \frac{P(T > t+s)}{P(T > t)} = \frac{\mathrm{e}^{-\lambda(t+s)}}{\mathrm{e}^{-\lambda t}} = \mathrm{e}^{-\lambda s} \\ &= P(T > s), \text{ as required.} \end{aligned}$$

(b) Weibull density: For $x \geq 0$,

$$\begin{aligned} f_X(x) &= \frac{d}{dx}\left(1 - \exp\left\{-\left(\frac{x}{\theta}\right)^{\beta}\right\}\right) = \beta\left(\frac{x}{\theta}\right)^{\beta-1}\left(\frac{1}{\theta}\right)\exp\left\{-\left(\frac{x}{\theta}\right)^{\beta}\right\} \\ &= \frac{\beta}{\theta^{\beta}}\, x^{\beta-1}\exp\left\{-\left(\frac{x}{\theta}\right)^{\beta}\right\} \end{aligned}$$

with $f_X(x) = 0$ for $x < 0$.

For Weibull distribution,

$$P(X > a+b \mid X > a) = \frac{P(X > a+b)}{P(X > a)} = \frac{\exp\left\{-\left(\frac{a+b}{\theta}\right)^{\beta}\right\}}{\exp\left\{-\left(\frac{a}{\theta}\right)^{\beta}\right\}} = \exp\left\{\frac{a^{\beta} - (a+b)^{\beta}}{\theta^{\beta}}\right\}$$

In the case $\beta = 2$,

$$\frac{P(X > a+b \mid X > a)}{P(X > b)} = \frac{\exp\left\{\frac{a^2 - (a+b)^2}{\theta^2}\right\}}{\exp\left\{-\left(\frac{b}{\theta}\right)^{2}\right\}} = \exp\left\{\frac{a^2 + b^2 - (a+b)^2}{\theta^2}\right\} = \exp\left\{-\frac{2ab}{\theta^2}\right\}$$

Ratio is not equal to 1, so distribution does not possess memoryless property.

Weibull distribution does possess the memoryless property when $\beta = 1$.

(c) For $\beta = 2$, $\theta = 1$, have

$$\begin{aligned} g(a) &= \exp\left\{a^2 - (a+b)^2\right\} = \exp\left\{-2ab - b^2\right\}, \\ h(a) &= \exp\left\{-2ab\right\}. \end{aligned}$$

Graphs:

Interpretation: Graph of $g(a)$ is decreasing in a, so the older the component is, the lower the probability that it will survive for a further time b, whatever the value of $b > 0$. Newer components are better, in terms of remaining lifetime distribution.

Graph of $h(a)$ is also decreasing in a, so the older the component, the greater the factor by which its survival probabilities are reduced compared with a new component.

3. (a) For g strictly increasing,

$$\begin{aligned} F_Y(y) &= P(Y \le y) = P(g(X) \le y) = P\left(X \le g^{-1}(y)\right) = F_X\left(g^{-1}(y)\right) \\ \Rightarrow f_Y(y) &= f_X\left(g^{-1}(y)\right) \frac{d}{dy} g^{-1}(y) \end{aligned}$$

For g strictly decreasing,

$$\begin{aligned} F_Y(y) &= P(Y \le y) = P(g(X) \le y) = P\left(X \ge g^{-1}(y)\right) = 1 - F_X\left(g^{-1}(y)\right) \\ \Rightarrow f_Y(y) &= -f_X\left(g^{-1}(y)\right) \frac{d}{dy} g^{-1}(y) \end{aligned}$$

In either case,

$$f_Y(y) = f_X\left(g^{-1}(y)\right) \left| \frac{d}{dy} g^{-1}(y) \right|.$$

(b) (i) Mean: $E[X] = \int_0^2 x^4/4 \, dx = \left[x^5/20\right]_0^2 = 2^5/20 = 32/20 = 8/5 = 1.6$.
Median m satisfies $0.5 = \int_0^m x^3/4 \, dx = \left[x^4/16\right]_0^m = m^4/16$, so $m^4 = 8$, $m = 8^{0.25} \approx 1.682$.
Mode is at $x = 2$.

(ii) For $0 < x < 2$,

$$F(x) = \int_{-\infty}^{x} f(u) \, du = \int_0^x \left(u^3/4\right) \, du = \left[u^4/16\right]_0^x = x^4/16$$

So in full,

$$F(x) = \begin{cases} 0 & x \le 0 \\ x^4/16 & 0 < x < 2 \\ 1 & x \ge 2 \end{cases}$$

(iii) With $Y = \sqrt{(X+2)/2}$, then for $y > 1$,

$$F_Y(y) = P(Y \le y) = P(\sqrt{(X+2)/2} \le y) = P(X \le 2y^2 - 2) = F_X(2(y^2 - 1))$$

$$\text{so that } F_Y(y) = \begin{cases} 0 & y \le 1 \\ (y^2 - 1)^4 & 1 < y < \sqrt{2} \\ 1 & y \ge \sqrt{2} \end{cases}$$

Differentiating,

$$f_Y(y) = \begin{cases} 8y(y^2 - 1)^3 & 1 < y < \sqrt{2} \\ 0 & \text{otherwise} \end{cases}$$

Range is $1 < Y < \sqrt{2}$.

$$
\begin{aligned}
E[Y] &= \int_1^{\sqrt{2}} 8y^2(y^2-1)^3\,dy = \int_1^{\sqrt{2}} 8y^2(y^6-3y^4+3y^2-1)\,dy \\
&= \int_1^{\sqrt{2}} (8y^8-24y^6+24y^4-8y^2)\,dy = \left[\frac{8y^9}{9}-\frac{24y^7}{7}+\frac{24y^5}{5}-\frac{8y^3}{3}\right]_1^{\sqrt{2}} \\
&= \left[8y^3\left(\frac{y^6}{9}-\frac{3y^4}{7}+\frac{3y^2}{5}-\frac{1}{3}\right)\right]_1^{\sqrt{2}} \\
&= 16\sqrt{2}\left(\frac{8}{9}-\frac{12}{7}+\frac{6}{5}-\frac{1}{3}\right)-8\left(\frac{1}{9}-\frac{3}{7}+\frac{3}{5}-\frac{1}{3}\right) \\
&= \frac{16\sqrt{2}\times 13}{315}+\frac{8\times 16}{315}\approx 1.3402
\end{aligned}
$$

4. (a) $\text{Cov}[X,Y] = E\left[(X - E[X])\,(Y - E[Y])\right]$
$\text{Corr}[X,Y] = \text{Cov}[X,Y]/\sqrt{\text{Var}[X]\text{Var}[Y]}$

Correlation values lie between -1 and +1; positive correlation indicates that the two variables tend to increase together, negative correlation that as one increases, the other decreases; the larger the absolute value of correlation, the stronger the linear relationship. Correlation +1 and -1 indicate a perfect linear relationship between the two variables; correlation 0 indicates no linear relationship.

(b) (i) Region of non-zero density:

(ii) Marginal density:

$$f_Y(y) \quad = \quad \int_{x=-y}^{y} \frac{\mathrm{e}^{-y}}{2y}\,dx = \left[\frac{x\mathrm{e}^{-y}}{2y}\right]_{x=-y}^{y} = \frac{y\mathrm{e}^{-y}}{2y} - \frac{-y\mathrm{e}^{-y}}{2y} = \mathrm{e}^{-y} \text{ for } y > 0$$

Conditional density:

$$f_{X|Y}(x \mid y) \quad = \quad \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{\mathrm{e}^{-y}/2y}{\mathrm{e}^{-y}} = \frac{1}{2y} \text{ for } -y < x < y$$

Distribution of Y is $\exp(1)$; conditional distribution of X is Uniform$(-y, y)$.

(iii) Expectations:

$$\begin{aligned} E[X] &= \int_{y=0}^{\infty}\int_{x=-y}^{y} x\,\frac{\mathrm{e}^{-y}}{2y}\,dx\,dy = \int_{y=0}^{\infty}\left[\frac{x^2\mathrm{e}^{-y}}{4y}\right]_{x=-y}^{y} dy \\ &= \int_{y=0}^{\infty}\left(\frac{y^2\mathrm{e}^{-y}}{2y} - \frac{y^2\mathrm{e}^{-y}}{2y}\right) dy = 0 \\ E[Y] &= \int_{y=0}^{\infty} y f_Y(y)\,dy = \int_{y=0}^{\infty} y\mathrm{e}^{-y}\,dy \\ &= \left[-y\mathrm{e}^{-y}\right]_{y=0}^{\infty} + \int_{y=0}^{\infty}\mathrm{e}^{-y}\,dy = 0 + \left[-\mathrm{e}^{-y}\right]_{y=0}^{\infty} = 1 \end{aligned}$$

(iv) Covariance:

$$\begin{aligned} E[XY] &= \int_{y=0}^{\infty}\int_{x=-y}^{y} xy\,\frac{\mathrm{e}^{-y}}{2y}\,dx\,dy = \int_{y=0}^{\infty}\left[\frac{x^2\mathrm{e}^{-y}}{4}\right]_{x=-y}^{y} dy \\ &= \int_{y=0}^{\infty}\left(\frac{y^2\mathrm{e}^{-y}}{4} - \frac{y^2\mathrm{e}^{-y}}{4}\right) dy = 0 \end{aligned}$$

so that $\text{Cov}[X,Y] = E[XY] - E[X]E[Y] = 0 - 0 \times 1 = 0$.

(v) X and Y are not independent; the range of possible X values depends upon the value of Y.

5. (a) $U = 2X + Y, V = 3X/Y$, so

$$\begin{aligned} X &= \frac{YV}{3} \Rightarrow U = \frac{2YV}{3} + Y = Y\left(\frac{2V}{3}+1\right) = \frac{Y}{3}(2V+3) \Rightarrow Y = \frac{3U}{2V+3} \\ \text{and so } X &= \frac{YV}{3} = \left(\frac{3U}{2V+3}\right)\left(\frac{V}{3}\right) = \frac{UV}{2V+3} \end{aligned}$$

Differentials:

$$\frac{\partial x}{\partial u} = \frac{v}{2v+3} \qquad \frac{\partial x}{\partial v} = \frac{u\left((2v+3)\times 1 - v \times 2\right)}{(2v+3)^2} = \frac{3u}{(2v+3)^2}$$

$$\frac{\partial y}{\partial u} = \frac{3}{2v+3} \qquad \frac{\partial y}{\partial v} = \frac{(2v+3)\times 0 - 3u \times 2}{(2v+3)^2} = \frac{-6u}{(2v+3)^2}$$

Jacobian:

$$J = \left(\frac{v}{2v+3}\right)\left(\frac{-6u}{(2v+3)^2}\right) - \left(\frac{3}{2v+3}\right)\left(\frac{3u}{(2v+3)^2}\right) = \frac{-6uv-9u}{(2v+3)^3} = \frac{-3u}{(2v+3)^2}$$

Density:

$$\begin{aligned} f_{U,V}(u,v) &= f_{X,Y}(x,y)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| \\ &= (2/\pi)\exp\left\{-\left(\left(\frac{uv}{2v+3}\right)^2 + \left(\frac{3u}{2v+3}\right)^2\right)\Big/ 2\right\}\left|\frac{-3u}{(2v+3)^2}\right| \\ &= \frac{6u}{\pi(2v+3)^2}\exp\left\{-\left(\frac{u^2\left(v^2+9\right)}{2(2v+3)^2}\right)\right\} \qquad u, v > 0 \end{aligned}$$

Region of positive density:

(b) Marginal density of V:

$$\begin{aligned} f_V(v) &= \frac{6}{\pi(2v+3)^2}\int_{u=0}^{\infty} u \exp\left\{-\left(\frac{u^2\left(v^2+9\right)}{2(2v+3)^2}\right)\right\} du \\ &= \frac{6}{\pi(2v+3)^2}\int_{u=0}^{\infty} u \exp\left\{-Au^2\right\} du \qquad \text{with } A = \frac{v^2+9}{2(2v+3)^2} \\ &= \frac{6}{\pi(2v+3)^2}\,\frac{1}{2A} = \frac{6}{\pi(2v+3)^2}\,\frac{(2v+3)^2}{v^2+9} = \frac{6}{\pi(v^2+9)} \qquad v > 0 \end{aligned}$$

6. (a) Differentiating,

$$M_X(t) = E[\mathrm{e}^{tX}] \qquad M'_X(t) = E[X\mathrm{e}^{tX}] \qquad M''_X(t) = E[X^2\mathrm{e}^{tX}]$$
$$\Rightarrow \quad M'_X(0) = E[X] \qquad M''_X(0) = E[X^2]$$

and so $\mathrm{Var}[X] = E[X^2] - (E[X])^2 = M''_X(0) - (M'_X(0))^2$, as required.

Repeated differentiation similarly gives $M_X^{(n)}(0) = E[X^n]$, where $M_X^{(n)}$ denotes the nth derivative of M_X.

With $Y = a + bX$,

$$M_Y(t) \;=\; E[\mathrm{e}^{tY}] = E[\mathrm{e}^{t(a+bX)}] = E[\mathrm{e}^{at} \times \mathrm{e}^{tbX}] = \mathrm{e}^{at} M_X(bt)$$

(b) The cumulants $\kappa_1, \kappa_2, \ldots$ are defined to be the coefficients in the power series

$$K_X(t) \;=\; \kappa_1 t + \frac{\kappa_2}{2!}\, t^2 + \frac{\kappa_3}{3!}\, t^3 + \cdots$$

κ_1 is equal to the mean; κ_2 is equal to the variance; κ_3 is third central moment; relationship between higher order cumulants and moments is more complicated.

(c) With $Z \sim N(0,1)$,

$$\begin{aligned} M_Z(t) \;&=\; E[\mathrm{e}^{tZ}] = \int_{-\infty}^{\infty} \mathrm{e}^{tz} \frac{1}{\sqrt{2\pi}}\, \mathrm{e}^{-z^2/2}\, dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{tz-(z^2/2)}\, dz \\ &=\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-((z-t)^2/2)+(t^2/2)}\, dz = \frac{\mathrm{e}^{t^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-(z-t)^2/2}\, dz \end{aligned}$$

Substituting $u = z - t$, so that $du = dz$, then

$$M_Z(t) \;=\; \frac{\mathrm{e}^{t^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-u^2/2}\, du = \mathrm{e}^{t^2/2} \int_{-\infty}^{\infty} f_Z(u)\, du = \mathrm{e}^{t^2/2}$$

since the standard normal density f_Z integrates to 1.

Now for $X = \mu + \sigma Z$, have $M_X(t) = \mathrm{e}^{\mu t} M_Z(\sigma t) = \mathrm{e}^{\mu t}\mathrm{e}^{(\sigma t)^2/2} = \mathrm{e}^{\mu t + (\sigma^2/2)t^2}$.

Hence $K_X(t) = \ln(M_X(t)) = \mu t + (\sigma^2/2)t^2$.

So in the case of the normal distribution we see that the first cumulant is equal to the mean, the second cumulant is equal to the variance, and all higher cumulants are zero.

In general, as stated in part (b), the first cumulant is the mean and the second cumulant is variance, but higher cumulants not generally zero, for non-normal distributions.

7. (a) For $U = \sqrt{X/n}$, have $X = nU^2$, so that $dx/du = 2nu$. Hence for $u \geq 0$,

$$\begin{aligned} f_U(u) &= f_X(x)\left|\frac{dx}{du}\right| = \frac{1}{2^{n/2}\,\Gamma\left(\frac{n}{2}\right)}\,(nu^2)^{\frac{n}{2}-1}\,\mathrm{e}^{-nu^2/2}\,|2nu| \\ &= \frac{n^{n/2}}{2^{(n/2)-1}\,\Gamma\left(\frac{n}{2}\right)}\,u^{n-1}\,\mathrm{e}^{-nu^2/2} \end{aligned}$$

as required.

(b) For $-\infty < t < \infty$, have

$$\begin{aligned} f_T(t) &= \int_0^\infty |u|\left(\frac{n^{n/2}}{2^{(n/2)-1}\,\Gamma\left(\frac{n}{2}\right)}\,u^{n-1}\,\mathrm{e}^{-nu^2/2}\right)\left(\frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-(ut)^2/2}\right)du \\ &= \frac{n^{n/2}}{2^{(n/2)-1}\,\Gamma\left(\frac{n}{2}\right)\,\sqrt{2\pi}}\int_0^\infty u^n\,\mathrm{e}^{-\left(n+t^2\right)u^2/2}\,du \end{aligned}$$

Substitute $w = u^2$, so that $dw = 2udu$, and then

$$\begin{aligned} f_T(t) &= \frac{n^{n/2}}{2^{(n/2)-1}\,\Gamma\left(\frac{n}{2}\right)\,\sqrt{2\pi}}\int_0^\infty w^{n/2}\,\mathrm{e}^{-\left(n+t^2\right)w/2}\,\frac{dw}{2\sqrt{w}} \\ &= \frac{n^{n/2}}{2^{n/2}\,\Gamma\left(\frac{n}{2}\right)\,\sqrt{2\pi}}\int_0^\infty w^{(n-1)/2}\,\mathrm{e}^{-\left(n+t^2\right)w/2}\,dw \end{aligned}$$

Set $\alpha = (n+1)/2$ and $\lambda = (n+t^2)/2$. Then

$$\begin{aligned} f_T(t) &= \frac{n^{n/2}}{2^{n/2}\,\Gamma\left(\frac{n}{2}\right)\,\sqrt{2\pi}}\int_0^\infty w^{\alpha-1}\,\mathrm{e}^{-\lambda w}\,dw \\ &= \frac{n^{n/2}}{2^{n/2}\,\Gamma\left(\frac{n}{2}\right)\,\sqrt{2\pi}}\,\frac{\Gamma(\alpha)}{\lambda^\alpha} \\ &= \frac{n^{n/2}}{2^{n/2}\,\Gamma\left(\frac{n}{2}\right)\,\sqrt{2\pi}}\,\frac{\Gamma\left(\frac{n+1}{2}\right)}{\left(\frac{n+t^2}{2}\right)^{(n+1)/2}} \\ &= \frac{n^{n/2}}{\sqrt{\pi}}\,\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}(n+t^2)^{-(n+1)/2} \\ &= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\,\sqrt{n\pi}}\left(1+\frac{t^2}{n}\right)^{-(n+1)/2} \end{aligned}$$