1. Suppose $Y_1, Y_2, Y_3, \ldots$ are independent Bernoulli random variables with respective success probabilities $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$. That is, for $i = 1, 2, \ldots$ the probability mass function of Y_i is given by

$$P(Y_i = 0) = 1 - \left(\frac{1}{2}\right)^i,$$
$$P(Y_i = 1) = \left(\frac{1}{2}\right)^i.$$

For $n = 1, 2, \ldots$ define $S_n = Y_1 + \cdots + Y_n$.

(a) For $n = 3$, compute the probability mass function of S_3. [6 marks]

(b) Find an expression for the probability $P(S_n = n)$ for $n = 1, 2, \ldots$ [2 marks]

(c) For large n, can the Central Limit Theorem be used to approximate the distribution of S_n? Explain your answer. [2 marks]

(d) Find $E[S_n]$ and $\text{Var}[S_n]$.

[You may use without proof the result that

$$\sum_{i=0}^{n} x^i = \frac{1 - x^{n+1}}{1 - x}.$$]

[6 marks]

(e) Defining the sample mean $\bar{Y}_n = S_n/n$, show that $\lim_{n\to\infty} E\left[\bar{Y}_n\right] = 0$ and find the value of $\lim_{n\to\infty} \text{Var}\left[\bar{Y}_n\right]$. Comment on why these limiting values are intuitively reasonable. [4 marks]

2. (a) Suppose the random variable T is exponentially distributed with parameter λ, so that T has distribution function

$$F_T(t) = \begin{cases} 0 & t < 0, \\ 1 - \mathrm{e}^{-\lambda t} & t \geq 0. \end{cases}$$

Define the *memoryless property*, explain the intuitive meaning of this property, and show that the distribution of T has the memoryless property. [4 marks]

(b) The random variable X is said to follow the Weibull distribution with parameters $\beta > 0$, $\theta > 0$ if the distribution function of X is given by

$$F_X(x) = \begin{cases} 0 & x < 0, \\ 1 - \exp\left\{-\left(\frac{x}{\theta}\right)^\beta\right\} & x \geq 0. \end{cases}$$

Derive an expression for the probability density function $f_X(x)$ of X. [2 marks]

Suppose the random variable X follows the Weibull distribution with parameters β, θ. For $a, b > 0$ find an expression for $P(X > a + b \mid X > a)$. [3 marks]

Show that in the case $\beta = 2$, the distribution of X does not possess the memoryless property. [3 marks]

For which values of the parameters β, θ, if any, does the Weibull distribution possess the memoryless property? [1 mark]

(c) Suppose that the lifetime Y (in years) of a particular electrical component follows the Weibull distribution with parameters $\beta = 2$, $\theta = 1$. Treating b as a fixed constant, sketch the graphs of the functions $g(a)$ and $h(a)$ defined by

$$g(a) = P(Y > a + b \mid Y > a),$$

$$h(a) = \frac{P(Y > a + b \mid Y > a)}{P(Y > b)}.$$

Give your intepretation of the shape of the graphs of $g(a)$ and $h(a)$. [7 marks]

3. (a) Suppose that X is a continuous random variable with probability density function $f_X(x)$, and that the random variable Y is defined by $Y = g(X)$ for some strictly monotonic function $g(x)$. Show that the probability density function of Y is given by

$$f_Y(y) \quad = \quad f_X\left(g^{-1}(y)\right)\left|\frac{d}{dy}\,g^{-1}(y)\right|.$$

[5 marks]

(b) Suppose X is a continuous random variable with density

$$f(x) \quad = \quad \begin{cases} x^3/4 & 0 < x < 2, \\ 0 & \text{otherwise.} \end{cases}$$

(i) Find the mean, median and mode of the distribution of X. [3 marks]

(ii) Find the cumulative distribution function $F(x)$ of X. [2 marks]

(iii) Find the cumulative distribution function and the probability density function of the random variable Y defined by $Y = \sqrt{(X+2)/2}$. What is the range of Y?

[7 marks]

Find the value of $E[Y]$. [3 marks]

4. (a) Give formulae defining the *covariance* and the *correlation* of two random variables X and Y. Explain how correlation values may be interpreted. [4 marks]

(b) Suppose that X, Y are continuous random variables with joint density function

$$f_{X,Y}(x,y) = \frac{e^{-y}}{2y}, \qquad y > 0, \quad -y < x < y.$$

(i) Draw the region of non-zero density. [2 marks]

(ii) Find the marginal density $f_Y(y)$ of Y and for $y > 0$ find the conditional density $f_{X|Y}(x \mid y)$ of X given $Y = y$. [5 marks]

For each of these two distributions, give the standard name of the distribution, specifying any parameter values. [2 marks]

(iii) Find $E[X]$ and $E[Y]$. [3 marks]

(iv) Find $E[XY]$ and hence find the covariance $\mathrm{Cov}[X,Y]$. [3 marks]

(v) Are X and Y independent? Explain your answer. [1 mark]

5. Suppose that X, Y are continuous random variables with joint density function

$$f_{X,Y}(x,y) = \begin{cases} (2/\pi)\exp\{-(x^2+y^2)/2\} & \text{for } x, y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Define the random variables U, V by

$$U = 2X + Y, \qquad V = 3X/Y.$$

(a) Find the joint density $f_{U,V}(u,v)$. Sketch the region where the random vector (U,V) has positive density. [13 marks]

(b) Find the marginal density of V. [7 marks]

[For part (b), you may use without proof the result that for $A > 0$,

$$\int_0^\infty u e^{-Au^2}\, du = \frac{1}{2A}. \quad]$$

6. (a) For any random variable X the moment generating function $M_X(t)$ is defined by $M_X(t) = E\left[e^{tX}\right]$.

Show that (i) $E[X] = M'_X(0)$ and (ii) $\text{Var}[X] = M''_X(0) - (M'_X(0))^2$, where $M'_X(t)$ and $M''_X(t)$ denote the first and second derivatives, respectively, of $M_X(t)$ with respect to t. [2 marks]

Write down an expression for $E[X^n]$, where $n = 1, 2, \ldots$, in terms of derivatives of the moment generating function $M_X(t)$. [1 mark]

Defining the random variable Y to be $Y = a + bX$, where a, b are non-random, derive an expression for the moment generating function $M_Y(t)$ in terms of M_X. [2 marks]

(b) For any random variable X, the cumulant generating function $K_X(t)$ is defined by $K_X(t) = \ln(M_X(t))$, where $M_X(t)$ is the moment generating function defined in part (a) above. Define the cumulants $\kappa_1, \kappa_2, \ldots$ of the distribution. [1 mark]

What can be said about the relationship between cumulants and moments of a distribution? [2 marks]

(c) Suppose Z follows the standard normal distribution, $Z \sim N(0,1)$, so that Z has probability density function

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \qquad -\infty < z < \infty.$$

Show that the moment generating function of Z is given by $M_Z(t) = e^{t^2/2}$. [7 marks]

Hence derive an expression for the moment generating function $M_X(t)$ of the normal random variable X with mean μ and variance σ^2, and a corresponding expression for the cumulant generating function $K_X(t)$ defined by $K_X(t) = \ln(M_X(t))$. [3 marks]

Based on your expression for $K_X(t)$, what can be said about the first, second, and higher cumulants of the $N(\mu, \sigma^2)$ distribution? To what extent do these statements generalise to distributions other than the normal distribution? [2 marks]

7. (a) Suppose the random variable X has the χ^2_n distribution for some positive integer n, so that the probability density function of X is given by

$$f_X(x) \quad = \quad \frac{1}{2^{n/2}\,\Gamma\left(\frac{n}{2}\right)}\, x^{\frac{n}{2}-1}\, \mathrm{e}^{-x/2} \qquad \text{for } x \geq 0.$$

Show that the random variable $U = \sqrt{X/n}$ has probability density function

$$f_U(u) \quad = \quad \frac{n^{n/2}}{2^{(n/2)-1}\,\Gamma\left(\frac{n}{2}\right)}\, u^{n-1}\, \mathrm{e}^{-nu^2/2} \qquad \text{for } u \geq 0.$$

[7 marks]

(b) Suppose that Y is a standard normal random variable, so that Y has probability density function

$$f_Y(y) \quad = \quad \frac{1}{\sqrt{2\pi}}\, \mathrm{e}^{-y^2/2} \qquad \text{for } -\infty < y < \infty.$$

Suppose further that Y is independent of the variable X of part (a) above, and define the random variable T by

$$T \quad = \quad \frac{Y}{\sqrt{X/n}}.$$

Derive an expression for the probability density function $f_T(t)$ of T.

[You may use without proof the results that

(i) the quotient $T = Y/U$ of independent continuous random variables Y, U has density function

$$f_T(t) \quad = \quad \int_{-\infty}^{\infty} |u|\; f_U(u)\; f_Y(ut)\; du;$$

(ii) for any $\alpha, \lambda > 0$,

$$\int_0^{\infty} x^{\alpha-1}\, \mathrm{e}^{-\lambda x}\, dx \quad = \quad \frac{\Gamma(\alpha)}{\lambda^{\alpha}}. \qquad]$$

[13 marks]