MATH764, Summer 2005. **Solutions**

1. [**Similar to the example discussed in class.**]

(a) Introduce a random variable X which represents the number of wrong connections in a day. Then X has a binomial distribution with parameters $n = 2000$ and $p = 0.001$. The required probability is

$$\binom{n}{0} p^0 (1-p)^n + \binom{n}{1} p^1 (1-p)^{n-1} + \binom{n}{2} p^2 (1-p)^{n-2}$$

$$= (.999)^{2000} + 2000 \cdot .001 \cdot (.999)^{1999} + \frac{2000 \cdot 1999}{2} (.001)^2 (.999)^{1998} = \mathbf{0.676676}.$$

(b) We use the Poisson approximation with $\lambda = np = 2$. We have

$$P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$$

$$e^{-\lambda} \left[1 + \lambda + \lambda^2/2\right] = e^{-2}[1 + 2 + 2] = \mathbf{0.676676}.$$

The Poisson approximation is very good; the first six decimal places coincide!

(c) Let now X represent the number of wrong connections in a day when the number of independent calls is n. We require to choose n such that

$$P(X \geq 1) \geq 0.9$$

or equivalently

$$P(X = 0) \leq 0.1.$$

If n is large we can approximate $P(X = 0)$ by $\exp(-pn) = \exp(-0.001n)$. Therefore we require the minimum n which satisfies

$$\exp(-0.001n) \leq 0.1,$$

or, equivalently,

$$\exp(0.001n) \geq 10.$$

Taking the logarithms of both sides, we obtain

$$n \geq (\ln 10)/0.001 = 2302.6$$

Thus the minimum number of independent calls required is **2303**.

(d) The reward from all 2000 connections equals $2000 \cdot 0.1 = 200$ pounds, but penalty for wrong connections equals $10E[X]$, where X is RV from (a). For binomial distribution, $E[X] = np = 2$. Thus, the average profit per day is $200 - 10 \cdot 2 = 180$ pounds.

2. **[Not seen but based on standard material.]**

(a) *First Pacific Inc.* will pay compensation higher than \$3 million if and only if $X = 5$ or $X = 4$. Therefore the required probability is

$$P(X = 5) + P(X = 4) = P(T < 1) + P(1 \leq T < 2)$$

$$= (1 - \exp(-0.5 \times 1) + (\exp(-0.5 \times 1) - \exp(-0.5 \times 2)) = 1 - \exp(-0.5 \times 2) \approx \mathbf{0.63}.$$

(b) To find the expected compensation we use the definition of the expected value:

$$EX = \sum_i x_i P(X = x_i) = 5P(X = 5) + 4P(X = 4) + 2P(X = 2) + 0P(X = 0).$$

Observe that

$$\begin{aligned} P(X = 5) &= P(T < 1) = 1 - \exp(-0.5 * 1) = 0.393 \\ P(X = 4) &= P(1 \leq T < 2) = \exp(-0.5 * 1) - \exp(-0.5 * 2) = 0.239 \\ P(X = 2) &= P(2 \leq T < 3) = \exp(-0.5 * 2) - \exp(-0.5 * 3) = 0.145 \end{aligned}$$

($P(X = 0)$ is not needed.) Therefore

$$EX = 5 * 0.393 + 4 * 0.239 + 2 * 0.145 = \mathbf{3.211}.$$

(c) Let $Y = f(X)$ be the amount of compensation *First Pacific Inc.* itself has to pay. The random variable Y has the following probability mass function:

Y	probab.
3	$P(X = 5) + P(X = 4) = 0.632$
2	$P(X = 2) = 0.145$
0	$P(X = 0)$ (not needed)

Therefore

$$EY = 3 * 0.632 + 2 * 0.145 = \mathbf{2.186}.$$

(d) Obviously, the expected payment by *Lloyds of London PLC.* equals the difference between the numbers calculated in parts (b) and (c). Therefore,

$$\text{Price } = 3.211 - 2.186 = 1.025.$$

3. **[Standard, similar problems were discussed in class.]**

(a) According to the definition of a uniform distribution,

$$f_X(x) = \begin{cases} \frac{1}{4}, & \text{if } x \in [0, 4]; \\ 0 & \text{otherwise .} \end{cases}$$

$$E[X] = \int_0^4 f_X(x) dx = 2.$$

(b) The range of X is $[0,4]$. Hence, the range of $Y = \sqrt{X}$ is $[0,2]$.

(c) First of all, the cumulative distribution function of X is

$$F_X(x) = \int_{-\infty}^{x} f_X(u)du = \begin{cases} 0, & \text{if } x < 0; \\ x/4, & \text{if } 0 \le x \le 4; \\ 1, & \text{if } x > 4. \end{cases}$$

Clearly, $F_Y(y) = 0$, if $y < 0$ and $F_Y(y) = 1$, if $y > 2$. Now for $0 \le y \le 2$, we have

$$F_Y(y) = P(Y \le y) = P(X \le y^2) = F_X(y^2) = y^2/4.$$

(d) Now, the density function is

$$f_Y(y) = \frac{dF_Y(y)}{dy}$$

$$= \begin{cases} \frac{y}{2}, & \text{if } y \in [0,2]; \\ 0 & \text{otherwise }. \end{cases}$$

(e) According to the definition,

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y)dy = \int_0^2 \frac{y^2}{2}dy = \frac{2^3}{6} = 4/3.$$

(f) Since X and Y are not independent, we cannot use formula $E[XY] = E[X] \times E[Y]$.

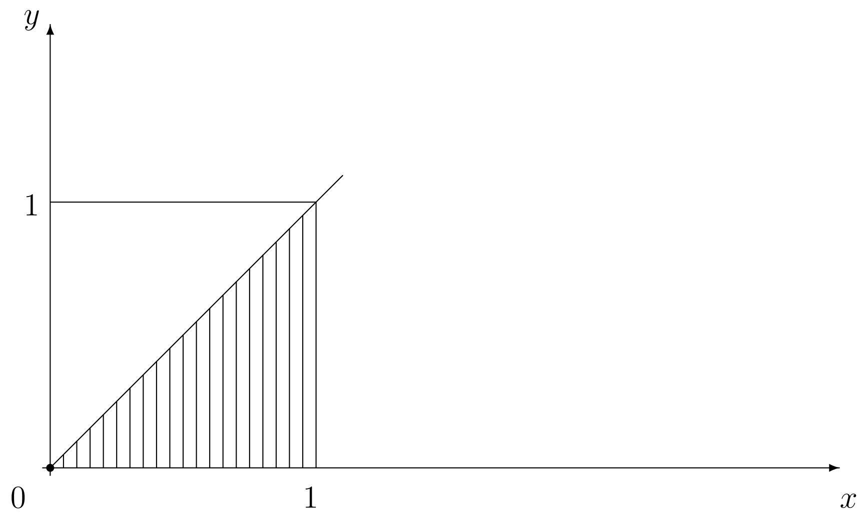
$$E[XY] = E[Y^3] = \int_0^2 y^3 \times \frac{y}{2}dy = \left.\frac{y^5}{10}\right|_0^2 = 3.2.$$

4. **[Similar to homework.]**

(a)

$$k\int_0^1 \left[\int_0^x (x+y)dy\right] dx = k\int_0^1 [xy + 0.5y^2]_0^x dx = k\int_0^1 \frac{3}{2}x^2 dx = \frac{3k}{2}\frac{1}{3} = \frac{k}{2}.$$

Hence, $k = 2$.



(b) Marginal density of X:

$$f_X(x) = \int_0^x 2(x+y)dy = 2[xy + 0.5 \times y^2]_0^x = 2 \times 1.5 \times x^2 = 3x^2, \qquad 0 \le x \le 1.$$

Marginal density of Y:

$$f_Y(y) = \int_y^1 2(x+y)dx = 2[0.5 \times x^2 + xy]_y^1 = 2[0.5 + y - 0.5 \times y^2 - y^2]$$

$$= 1 + 2y - 3y^2, \qquad 0 \le y \le 1.$$

(c)

$$f_{Y|X}(y|x) = \frac{2(x+y)}{3x^2}, \qquad 0 \le y \le x \le 1.$$

(d)

$$f_{Y|X}(y|1) = \frac{2(1+y)}{3}, \qquad 0 \le y \le 1.$$

So,

$$P(Y > 1/3|X = 1) = \frac{2}{3}\int_{1/3}^1 (1+y)dy = \frac{2}{3}[y + 0.5 \times y^2]_{1/3}^1 = \frac{2}{3}[1 + 1/2 - 1/3 - 1/18] = \frac{20}{27}.$$

5. **[Similar to homework.]**

$$\int_0^\infty \int_0^\infty f(x,y)dxdy = k\int_0^\infty e^{-x}dx \int_0^\infty e^{-xy}xdy = k\int_0^\infty e^{-x}dx \times 1 = k,$$

so, $k = 1$.

We must find the inverse transformation.

$$v = \ln(y+1); \quad e^v = y+1; \quad y = e^v - 1; \quad x = u - y = u - e^v + 1.$$

Thus,

$$\begin{cases} x = u - e^v + 1; \\ y = e^v - 1, \end{cases}$$

where $u - e^v + 1 \geq 0$ and $e^v - 1 \geq 0$, that is $v \geq 0$ and $u \geq e^v - 1$.

The Jacobian of this transformation is

$$J = det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = det \begin{bmatrix} 1 & -e^v \\ 0 & e^v \end{bmatrix} = e^v.$$

Thus,

$$f_{U,V}(u, v) = f(u - e^v + 1, e^v - 1)e^v = (u - e^v + 1)e^{-(u-e^v+1)e^v}e^v$$

$$= e^v(u - e^v + 1)\exp\{-e^v(u - e^v + 1)\},$$

where $v \geq 0$, $u \geq e^v - 1$.

6. **[Bookwork and similar to homework.]**

(a) Suppose $X_1, X_2, \ldots$ is a sequence of iid RVs with mean μ and variance σ^2. Let Φ denote the standard normal CDF. Then, for any real number x

$$P\left\{\frac{\sum_{i=1}^{n}(X_i - \mu)}{\sigma\sqrt{n}} \leq x\right\} \longrightarrow \Phi(x) \text{ as } n \to \infty.$$

(b) (i)

$$\mu = E[X_i] = 0.5 \times (-2) + 0.1 \times 1 = -0.9;$$

$E[X_i^2] = 0.5 \times 4 + 0.1 \times 1 = 2.1$; $\sigma^2 = Var[X_i] = 2.1 - (0.9)^2 = 1.29$, so $\sigma = 1.136$.

Let $S = \sum_{i=1}^{100} X_i$. Then

$$P(S \leq -70) = P\left(\frac{S - n\mu}{\sigma\sqrt{n}} \leq \frac{-70 - 100 \times (-0.9)}{10 \times 1.136}\right) \approx P(Z \leq 1.76) \approx 0.9608.$$

(Here Z has standard normal distribution.)

(ii) Denote $Y_i = X_i^2$. The PMF of Y_i is $P(Y_i = 4) = 0.5$; $P(Y_i = 0) = 0.4$; $P(Y_i = 1) = 0.1$.

$$\mu = E[Y_i] = 0.5 \times 4 + 0.1 \times 1 = 2.1$$

$E[Y_i^2] = 0.5 \times 16 + 0.1 \times 1 = 8.1$; $\sigma^2 = Var[Y_i] = 8.1 - (2.1)^2 = 3.69$, so $\sigma = 1.921$.

Let $S = \sum_{i=1}^{100} X_i^2 = \sum_{i=1}^{100} Y_i$. Then

$$P(S \geq 200) = P\left(\frac{S - n\mu}{\sigma\sqrt{n}} \geq \frac{200 - 100 \times (2.1)}{10 \times 1.921}\right)$$

$$\approx P(Z \geq -0.521) = P(Z \leq +0.521) \approx 0.6985.$$

(Here Z has standard normal distribution.)

(c) Set $S_n = \sum_{i=1}^n X_i^2$

$$0.99 = P(S_n \geq 200) \approx P\left(Z \geq \frac{200 - 2.1n}{1.921\sqrt{n}}\right) = P\left(Z \leq \frac{2.1n - 200}{1.921\sqrt{n}}\right).$$

The 99% critical value is 2.33, so we must solve for n

$$2.1n - 200 = 2.33 \times 1.921\sqrt{n}.$$

Set $n = x^2$. The equation above becomes

$$2.1x^2 - 4.476x - 200 = 0.$$

So,

$$x = \frac{4.476 + \sqrt{4.476^2 + 800 \times 2.1}}{2 \times 2.1} = 10.88.$$

(The other root is negative.) Hence, $n = 119$ is the smallest integer $\geq x^2$.

7. **[Similar to homework.]**

(a) $u = g(x) = \sqrt{\frac{x}{n}}$, so $x = g^{-1}(u) = nu^2$ and $\frac{dx}{du} = 2nu$. Thus,

$$f_U(u) = f_X(g^{-1}(u))\frac{d}{du}[g^{-1}(u)] = \frac{1}{2^{n/2}\Gamma(n/2)}(nu^2)^{n/2-1}e^{-nu^2/2}2nu$$

$$= \frac{1}{2^{n/2-1}\Gamma(n/2)}n^{n/2}(u^2)^{\frac{n-1}{2}}e^{-nu^2/2}.$$

(b)

$$f(t) = \int_{-\infty}^{\infty} |u| f_U(u) f_Y(ut) du \text{ (since } f_U(u) = 0 \text{ if } u < 0)$$

$$= \int_0^\infty u f_U(u) f_Y(ut) du = \int_0^\infty \frac{1}{2^{n/2-1}\Gamma(n/2)}n^{n/2}(u^2)^{n/2}e^{-nu^2/2}\frac{1}{\sqrt{2\pi}}e^{-u^2t^2/2}du$$

$$= \frac{n^{n/2}}{2^{n/2-1}\Gamma(n/2)\sqrt{2\pi}}\int_0^\infty (u^2)^{n/2}e^{-(n+t^2)u^2/2}du$$

(Set $u^2 = x$; $2udu = dx$; $du = \frac{dx}{2\sqrt{x}} = \frac{1}{2}x^{-1/2}dx$.)

$$= \frac{n^{n/2}}{2^{\frac{n-1}{2}}\sqrt{\pi}\Gamma(n/2)}\int_0^\infty x^{n/2}e^{-\frac{(n+t^2)x}{2}}\frac{1}{2}x^{-1/2}dx$$

$$= \frac{n^{n/2}}{2^{\frac{n+1}{2}}\sqrt{\pi}\Gamma(n/2)}\int_0^\infty x^{(\frac{n+1}{2})-1}e^{-\frac{(n+t^2)x}{2}}dx$$

(We have obtained the Gamma integral with $\alpha = \frac{n+1}{2}$ and $\lambda = \frac{1}{2}(n+t^2)$.)

$$= \frac{n^{n/2}}{2^{\frac{n+1}{2}}\sqrt{\pi}\Gamma(n/2)}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\left[\frac{1}{2}(n+t^2)\right]^{\frac{n+1}{2}}} = \frac{n^{-1/2}n^{n/2+1/2}\Gamma\left(\frac{n+1}{2}\right)}{2^{n/2+1/2}\left(\frac{1}{2}\right)^{n/2+1/2}(n+t^2)^{n/2+1/2}\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)}$$

$$= \frac{1}{\sqrt{n\pi}}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\left(1 + \frac{t^2}{n}\right)^{-\left(\frac{n+1}{2}\right)}.$$