1. In an automatic telephone exchange the probability that any one call is wrongly connected is 0.001.

(a) For a day when 2000 independent calls are connected, determine the probability that at most 2 wrong connections are made using the exact formula. (The answer must be presented with precision to 6 decimal places.) [6 marks]

(b) Use the Poisson distribution to find an approximate value of the probability asked for in part (a), with precision to 6 decimal places. Comment on the accuracy of the Poisson approximation. [4 marks]

(c) What is the minimum number of independent calls required before there is a probability of 0.9 that at least one of the calls is wrongly connected. [8 marks]

(d) Suppose the owner of this exchange receives an income of 10 pence for each and every connection, but a wrong connection costs him 10 pounds. What is the average profit per day with 2000 independent calls? [2 marks]

2. An expensive piece of equipment is insured with the insurance company *First Pacific Inc.* according to the following table in which T denotes the first time (in years) that the equipment fails to function within specified parameters; X denotes the amount (in millions of pounds) *First Pacific Inc.* will have to pay to the owners of the equipment:

$$\begin{array}{lll} X = 5 & \text{if} & T < 1 \\ X = 4 & \text{if} & 1 \le T < 2 \\ X = 2 & \text{if} & 2 \le T < 3 \\ X = 0 & \text{if} & T \ge 3 \end{array}$$

Suppose T is an exponential random variable defined by

$$P(T > t) = \exp(-0.5t), \ t \ge 0.$$

(a) Find the probability that *First Pacific Inc.* will pay compensation higher than £3 million. [3 marks]

(b) Find the expected compensation *First Pacific Inc.* will pay. [9 marks]

(c) Suppose *First Pacific Inc.* buys a policy from *Lloyds of London PLC.* to protect itself against very large claims. According to this policy, if *First Pacific Inc.* agrees to settle a claim of £x million, it will, in fact, pay only £$f(x)$ million, where $f(x) \le x$, and the rest will be paid by *Lloyds of London PLC.*. Suppose that

$$f(x) = x \ \text{ if } \ x < 3, \quad f(x) = 3 \ \text{ if } \ x \ge 3.$$

Find the expected compensation *First Pacific Inc.* will pay assuming it has purchased the policy described in this part of the question. [6 marks]

(d) *Lloyds of London PLC.* agrees to sell this policy for the price equal to their expected payment. Calculate this price. [2 marks]

3. Let $X \sim U[0,4]$ be a uniform on $[0,4]$ random variable and $Y = \sqrt{X}$.

(a) Write down the probability density function of X and mathematical expectation $E[X]$. [2 marks]

(b) Determine the range of Y. [1 marks]

(c) Find the cumulative distribution function of Y. [7 marks]

(d) Find the probability density function of Y. [3 marks]

(e) Find the mathematical expectation of Y. [4 marks]

(f) Find the mathematical expectation of the product XY. [3 marks]

4. Suppose the random variables X and Y are jointly continuous with joint density function

$$f(x,y) = k(x+y), \quad 0 \le y \le x \le 1.$$

(a) Draw the region where the density is positive and determine the value of the constant k. [4 marks]

(b) Find the marginal densities of X and Y. [9 marks]

(c) Find the conditional density of Y given $X = x,\ 0 \le x \le 1$. [2 marks]

(d) Calculate $P(Y > \frac{1}{3} | X = 1)$. [5 marks]

5. Suppose the random variables X and Y are jointly continuous with joint density function

$$f(x,y) = kxe^{-x(y+1)}, \qquad x, y \ge 0.$$

Determine the value of the constant k. [3 marks]

Let the random variables U and V be defined by

$$U = X + Y, \quad V = \ln(Y+1).$$

Find the joint density of U and V, and indicate the range of the random vector (U, V). [17 marks]

6. The random variables $X_1, \ldots, X_{100}$ are independent and identically distributed, each with probability mass function

$$p(-2) = 0.5, \quad p(0) = 0.4, \quad p(1) = 0.1.$$

(a) State the Central Limit Theorem. [3 marks]

(b) Using the Central Limit Theorem, find approximations for

(i) $P(\sum_{i=1}^{100} X_i \le -70)$ [5 marks]

(ii) $P(\sum_{i=1}^{100} X_i^2 \ge 200)$ [5 marks]

(c) Consider now a sample $X_1, \ldots, X_n$, of size n. How large must n be to ensure that the sum of squares, $\sum_{i=1}^{100} X_i^2$, is greater than 200 with probability greater than 0.99? [7 marks]

7. Suppose the random variable X has a χ_n^2 distribution, i.e. it has density function

$$f_X(x) = \frac{1}{2^{n/2}\Gamma(\frac{n}{2})} x^{\frac{n}{2}-1} e^{-x/2}, \quad x > 0.$$

(a) Show that the random variable $U = \sqrt{X/n}$ has density function

$$f_U(u) = \frac{1}{2^{(n/2)-1}\Gamma(\frac{n}{2})} n^{(n/2)} (u^2)^{\frac{n-1}{2}} e^{-nu^2/2}.$$

[7 marks]

(b) Suppose Y is a standard normal random variable which is independent of X. Using the result of part (a) show that the t–statistic

$$T = \frac{Y}{\sqrt{X/n}}$$

has density function

$$f_T(t) = \frac{\Gamma(\frac{n+1}{2})}{\sqrt{n\pi}\Gamma(\frac{n}{2})} \left(1 + \frac{t^2}{n}\right)^{-\frac{(n+1)}{2}}.$$

[13 marks]

Hint: Recall that the quotient Y/U of independent continuous random variables Y and U has density function

$$f(t) = \int_{-\infty}^{\infty} |u| f_U(u) f_Y(ut) du.$$

The gamma density is

$$f_{\alpha,\lambda}(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \; x > 0, \qquad \alpha, \lambda > 0.$$

[NORMAL DISTRIBUTION TABLE]