MATH764, Summer 2004. **Solutions**

1. **[Finite probability models and the binomial distribution were discussed in depth.]**

(a) Obviously, $p = 1/3$ is the probability that A wins a round. (That happens only if $X = 2$.) The expected gain for A equals $(1/3)(a+b)$, and we must solve the following equation with respect to b

$$a = \frac{1}{3} \times (a+b)$$

given $a = 1$. Now $b = 2a = 2$.

(b) Let Z be the number of the rounds, out of 5, which are in favour of A. Then $Z \sim Bin(5, 1/3)$. Probability that A receives the prize equals

$$P\{Z \geq 3\} = P\{Z=3\}+P\{Z=4\}+P\{Z=5\} = \frac{10\cdot 4}{3^5}+\frac{5\cdot 2}{3^5}+\frac{1}{3^5} = \frac{17}{81} \approx 0.2099.$$

Hence, for the given value $a = 1$ equation $a = \frac{17}{81}(a+b)$ results in $b = \frac{64}{17} \approx 3.7647$.

(c) Let us continue the game with 3 fictitious rounds. B will win the prize only if he wins all these rounds. Since $P\{\text{B wins a round}\} = \frac{2}{3}$, the chance for B to win the prize equals $\left(\frac{2}{3}\right)^3 = \frac{8}{27}$, and the prize must be divided proportionally to the chances in the ratio 8:19.

Finally, B receives

$$\frac{8}{27}(a+b) = \frac{8}{27}\left(1+\frac{64}{17}\right) = \frac{24}{17} \approx 1.4118$$

and A receives

$$\frac{19}{27}(a+b) = \frac{57}{17} \approx 3.3529.$$

(d) According to the CLT, the number of rounds, V, that the player A wins, satisfies the approximate relation

$$P\{V \leq x\} = P\left\{\frac{V-np}{\sqrt{np(1-p)}} \leq \frac{x-np}{\sqrt{np(1-p)}}\right\} \approx \Phi\left(\frac{x-np}{\sqrt{np(1-p)}}\right),$$

where $p = 1/3$.

If $x = n/2$ then probability that the player A will lose the whole game approximately equals

$$P\{V \leq n/2\} \approx \Phi\left(\frac{n/6}{\sqrt{n\cdot 2/9}}\right) = \Phi\left(\frac{\sqrt{n}}{2\sqrt{2}}\right) \to 1 \quad \text{when} \quad n \to \infty.$$

thus probability that A wins the prize goes to 0.

2. **[Similar to problems discussed in class.]**

(a) $G(s) = \sum_{k=0}^{\infty} s^k \frac{\lambda^k}{k!} e^{-\lambda}$, by the definition. Now

$$G(s) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(s\lambda)^k}{k!} = e^{-\lambda} \cdot e^{s\lambda} = e^{\lambda(s-1)}.$$

(b) We know that $P\{X = i\} = \frac{\lambda^i}{i!} e^{-\lambda}$ and $P\{Y = i\} = \frac{\lambda^i}{i!} e^{-\lambda}$, $i = 0, 1, 2, \ldots$ Using the theorem on the sum of RVs we obtain

$$P\{Z = k\} = \sum_{i=0}^{k} P\{X = i\} P\{Y = k - i\} = \sum_{i=0}^{k} \frac{\lambda^i}{i!} e^{-\lambda} \frac{\lambda^{k-i}}{(k-i)!} e^{-\lambda}$$

$$= e^{-2\lambda} \frac{\lambda^k}{k!} \sum_{i=0}^{k} \frac{k!}{i!(k-i)!} = e^{-2\lambda} \frac{\lambda^k}{k!} 2^k = \frac{(2\lambda)^k}{k!} e^{-2\lambda}.$$

Here we have used the equality $\sum_{i=0}^{k} \frac{k!}{i!(k-i)!} = 2^k$ which follows, for instance, from the total sum of the binomial probabilities:

$$\sum_{i=0}^{k} \frac{k!}{i!(k-i)!} \left(\frac{1}{2}\right)^i \left(\frac{1}{2}\right)^{k-i} = 1.$$

Therefore, Z has the Poisson distribution with parameter 2λ.

The students who use generating functions get the full credit.

(c) The number of particles ("events") during a fixed time interval is usually Poisson distributed. Hence $X \sim Poisson(1000)$, because $E[X] = \lambda$ coincides with the parameter of the Poisson distribution.

(d) Since $\lambda = 1000$ is big we can use the normal approximation:

$$P\{900 < X < 1000\} = P\left\{\frac{900 - 1000}{\sqrt{1000}} < \frac{X - 1000}{\sqrt{1000}} < 0\right\}$$

$$\approx P\{-3.162 < Z < 0\} = \Phi(3.162) - 0.5 \approx 0.4992.$$

(Here Z is a standard normal RV.)

3. **[Standard.]**

The CDF is defined by $F(x) = P(X \leq x)$, the density by

$$P(a < X \leq b) = \int_a^b f(x)dx.$$

Relationships:

$$f(x) = F'(x), \qquad F(x) = \int_{-\infty}^{x} f(u)du.$$

(a)

$$\int_0^{\pi} \sin x dx = [-\cos\ x]_0^{\pi} = 2, \text{ so } K = \frac{1}{2}.$$

(b)

$$F(x) = \int_0^x \frac{1}{2}\sin u du = \frac{1}{2}[-\cos u]_0^x = \frac{1}{2}[-\cos x + 1].$$

$$F(x) = \frac{1}{2}(1 - \cos x), \qquad 0 < x < \pi.$$

(c) The range of Y is $(0, \sqrt{\pi})$.

$$F_Y(y) = P(Y \leq y) = P(\sqrt{X} \leq y) = P(X \leq y^2) = F(y^2) = \frac{1}{2}(1 - \cos(y^2)).$$

$$F_Y(y) = \frac{1}{2}(1 - \cos(y^2)), \qquad 0 < y < \sqrt{\pi}.$$

Density:

$$f_Y(y) = F_Y'(y) = \frac{1}{2}\sin(y^2)2y = y\sin(y^2), \quad 0 < y < \sqrt{\pi}.$$

One can also use the transformation method.

4. **[Similar to homework.]**

(a) Focus first on X. The cdf is

$$F_X(x) = 1 - P(X > x) = 1 - e^{-\lambda x}.$$

Therefore the density is

$$f_X(x) = \frac{d}{dx}F_X(x) = \lambda e^{-\lambda x}.$$

In the same way we obtain

$$f_Y(y) = \mu e^{-\mu y}.$$

By independence the joint density is

$$f(x, y) = \lambda\mu e^{-(\lambda x + \mu y)}.$$

(b) The inverse transformation is

$$x = r\cos\theta, \quad y = r\sin\theta.$$

The Jacobian is

$$det \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{bmatrix} = det \begin{bmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix} = r.$$

Applying now the formula for the transformed density we obtain

$$f_{R\Theta}(r, \theta) = f(r\cos\theta, r\sin\theta)r$$

$$= \lambda\mu r e^{-(\lambda r\cos\theta + \mu r \sin\theta)}, \quad r > 0,\ 0 \le \theta \le \pi/2.$$

(c) The marginal density of Θ is

$$f_\Theta(\theta) = \int_0^\infty f_{R\Theta}(r,\theta)dr = \lambda\mu \int_0^\infty r e^{-r(\lambda\cos\theta + \mu\sin\theta)}dr.$$

Set $\alpha = \lambda\cos\theta + \mu\sin\theta$ and use the formula given in the problem to obtain

$$f_\Theta(\theta) = \frac{\lambda\mu}{(\lambda\cos\theta + \mu\sin\theta)^2}.$$

(d) If $\lambda = \mu$ then

$$f_\Theta(\theta) = \frac{1}{(\cos\theta + \sin\theta)^2} = \frac{1}{1+\sin 2\theta}.$$

Now

$$E[\Theta] = \int_0^{\pi/2} \frac{\theta\ d\theta}{1+\sin 2\theta} = \frac{1}{4}\int_0^\pi \frac{x\ dx}{1+\sin x}$$

$$= \frac{1}{4}[-\pi\tan(-\pi/4) + 2\ln|\cos(-\pi/4)| - 2\ln|\cos(\pi/4)|] = \frac{\pi}{4}.$$

The students who notice that $f_\Theta(\theta)$ is symmetric on $[0,\pi/2]$ and deduce that $E[\Theta] = \pi/4$ will be awarded full credit.

5. **[Bookwork and similar to homework.]**

(a) The moment generating function of X is

$$M_X(t) = Ee^{tX} = \sum_{k=0}^\infty p(k)e^{tk}$$

provided the expectation exists.

(b) Using this definition, we have

$$M_Y(t) = \sum_{k=0}^\infty \left(\frac{1}{2}\right)^{k+1} e^{tk} = \frac{1}{2}\sum_{k=0}^\infty \left(\frac{e^t}{2}\right)^k = \frac{1}{2} \times \frac{1}{1-\frac{e^t}{2}} = \frac{1}{2-e^t},$$

where we used the formula

$$\sum_{k=0}^\infty s^k = \frac{1}{1-s}, \quad |s| < 1$$

with

$$s = \frac{e^t}{2}.$$

This means that the admissible values of t are such that

$$\frac{e^t}{2} < 1, \quad \text{i.e.} \quad t < \ln 2.$$

(c) To find EY, we use the formula

$$EY = M'_Y(t)|_{t=0}.$$

We have

$$M'_Y(t) = \left[\frac{1}{2-e^t}\right]' = \frac{e^t}{(2-e^t)^2},$$

so

$$EY = \frac{e^0}{(2-e^0)^2} = \frac{1}{(2-1)^2} = 1.$$

To find EY^2, we use $EY^2 = M''_Y(t)|_{t=0}$. We have

$$M''_Y(t) = \frac{d}{dt}[e^t(2-e^t)^{-2}] = e^t(2-e^t)^{-2} + 2e^{2t}(2-e^t)^{-3}.$$

Hence

$$EY^2 = 1 + 2 = 3.$$

(d) Therefore, $Var\ Y = EY^2 - (EY)^2 = 3 - 1 = 2.$

6. **[Parts (a) and (b) fairly standard, (c) similar to bookwork and homework.]**

(a) $\mu = -1 \cdot \frac{1}{2} + 3 \cdot \frac{1}{2} = 1;\quad EX^2 = \frac{1}{2} + \frac{9}{2} = 5;\quad \sigma^2 = Var\ X = EX^2 - (EX)^2 = 4.$

(b)

$$P\{\ln(\bar{X}) > 0\} = P\{\bar{X} > 1\} = P\{\sum_{i=1}^{200} X_i > 200\}$$

$$= P\left\{\frac{\sum_{i=1}^{200} X_i - 200 \cdot 1}{2\sqrt{200}} > \frac{200 - 200 \cdot 1}{2\sqrt{200}}\right\} \approx P\{Z > -0\} = 0.5$$

(Here Z has standard normal distribution.)

(c)

$$P\left\{\sum_{i=1}^{n} X_i > 190\right\} \approx P\left\{Z > \frac{190 - n \cdot 1}{2\sqrt{n}}\right\} > 0.99.$$

The 99% critical value is 2.33, so we want to find n such that

$$\frac{190 - n}{2\sqrt{n}} < -2.33.$$

Set $n = x^2$ and solve equation $190 - x^2 = -4.66\ x$, i.e.

$$x^2 - 4.66x - 190 = 0.$$

$\Delta = 4.66^2 + 4 \cdot 190 = 781.72;\ \sqrt{\Delta} = 27.96;$

$$x_1 = \frac{4.66 + 27.96}{2} = 16.31; \qquad x_2 = \frac{4.66 - 27.96}{2} = -11.65.$$

Thus, $n_1 = 266.02$; $n_2 = 135.72$. Clearly, $190 - n_2 > 0$ so n_2 does not satisfy our requirement. Thus the minimum n is 267.

7. **[Bookwork.]** (a) $\Gamma(t) = \int_0^\infty u^{t-1} e^{-u} du$, $(t > 0)$ by definition.

$$\Gamma(1) = \Gamma(2) = 1; \quad \Gamma(\frac{1}{2}) = \sqrt{\pi};$$

$$\Gamma(t+1) = t\Gamma(t), \qquad t > 0; \qquad \Gamma(n) = (n-1)!, \text{ if } n > 0 \text{ is integer.}$$

(b) Definition. If Z is a standard normal random variable, the distribution of $U = Z^2$ is called chi-square distribution with 1 degree of freedom. Notation: $U \sim \chi_1^2$. If $U_1, U_2, \ldots, U_m$ are independent χ_1^2 random variables, the distribution of $U = U_1 + U_2 + \ldots + U_m$ is called the chi-square distribution with m degrees of freedom. Notation: $U \sim \chi_m^2$.

(c) We know that density of V is

$$f_V(v) = \frac{\left(\frac{1}{2}\right)^{n/2}}{\Gamma\left(\frac{n}{2}\right)} v^{n/2-1} e^{-v/2}.$$

Consider $A = V/n$. Using transformation method

$$f_A(a) = f_V(na)n = \frac{\left(\frac{1}{2}\right)^{n/2}}{\Gamma\left(\frac{n}{2}\right)} (na)^{n/2-1} e^{-na/2} n.$$

Similarly, setting $B = U/m$, we obtain

$$f_B(b) = \frac{\left(\frac{1}{2}\right)^{m/2}}{\Gamma\left(\frac{m}{2}\right)} (mb)^{m/2-1} e^{-mb/2} m.$$

Since A and B are independent, we can use the formula for the density of the quotient $W = B/A$:

$$f_W(w) = \int_0^\infty |a| f_A(a) f_B(wa) da$$

$$= \int_0^\infty a \frac{\left(\frac{1}{2}\right)^{\frac{n+m}{2}}}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)} (na)^{n/2-1} n (mwa)^{m/2-1} m e^{-na/2} e^{-mwa/2} da$$

$$= \frac{\left(\frac{1}{2}\right)^{\frac{n+m}{2}}}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)} n^{n/2-1} (mw)^{m/2-1} nm \int_0^\infty a^{n/2+m/2-1} e^{-\frac{1}{2}(n+mw)a} da.$$

Note that Gamma density is

$$f_{\alpha,\lambda}(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}.$$

In our case $\lambda = \frac{1}{2}(n + mw)$ and $\alpha = \frac{n+m}{2}$ and a plays the role of x. So

$$f_W(w) = \frac{\left(\frac{1}{2}\right)^{\frac{n+m}{2}}}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)} n^{n/2} m^{m/2} w^{\frac{m}{2}-1} \frac{\Gamma\left(\frac{m+n}{2}\right)}{\left[\frac{1}{2}(n + mw)\right]^{\frac{n+m}{2}}}$$

$$= \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)} \left(\frac{m}{n}\right)^{m/2} w^{\frac{m}{2}-1} \left(1 + \frac{m}{n} w\right)^{-\frac{n+m}{2}}.$$