1. Let X be the uniform discrete RV taking values 0, 1, and 2 equiprobably, and $Y = X + 1 \pmod 3$, i.e. $Y = X + 1$ if $X < 2$ and $Y = 0$ if $X = 2$.

Two players A and B pay the entrance fees £a and £b correspondingly and play the following game: if $X > Y$ then A receives all the money ($a + b$ pounds); if $X < Y$ then B receives all the money. Let $a = 1$.

(a) Calculate the fair value of b, such that the entrance fee equals expected gain for each participant.

(b) A modification of this game is as follows. The random pairs (X, Y) described above are realized in $n = 5$ independent rounds, and A receives $(a + b)$ pounds if he wins 3 or more rounds; similarly B receives $(a + b)$ pounds if he wins 3 or more rounds. Calculate the fair value of b, if $a = 1$, as previously.

(c) Let $a = 1$ and b be as in item (b). The game described in (b) began, A won the first two rounds, and the players had to stop playing, before the game actually finished. What is the fair division of the prize $(a + b)$?

(d) Let n be a very big odd number and the prize in the game (b) goes to the person who wins more than a half of the rounds. Using the Central Limit Theorem, show that probability of the prize to go to the player A, approaches 0 as $n \to \infty$.

2. (a) Derive the Probability Generating Function for a Poisson(λ) random variable.

(b) Let X and Y be two independent Poisson random variables with the same parameter $\lambda > 0$. Find the Probability Mass Function for $Z = X + Y$.

(c) Let X be the number of particles registered by a Geiger counter during 1/2 hour. It is known that $E[X] = 1000$. Suggest a proper probability distribution for X.

(d) Using the Central Limit Theorem, evaluate $P\{900 < X < 1000\}$, where X is the random variable described in (c).

3. Suppose X is a continuous random variable. State the definition of the Cumulative Distribution Function and the density function and the relationship between them (no more than 80 words).

Suppose X is a continuous random variable with density

$$f(x) = K \sin(x), \quad 0 < x < \pi.$$

(a) Find the constant K.

(b) Find the Cumulative Distribution Function of X.

(c) Find the density and the Cumulative Distribution Function of $Y = \sqrt{X}$. What is the range of Y?

4. In a physical experiment particles hit a plane at random. The coordinates X and Y of the impact points can be viewed as independent exponential random variables defined by

$$P(X > x) = e^{-\lambda x},\ x \geq 0, \qquad P(Y > y) = e^{-\mu y},\ y \geq 0.$$

(a) Find the marginal densities of X and Y and the joint density of X and Y.

(b) Consider the polar coordinates

$$R = \sqrt{X^2 + Y^2}, \quad \Theta = \tan^{-1}\left(\frac{Y}{X}\right).$$

Find the joint density of R and Θ. Indicate the range of R and Θ.

(c) Find the marginal density of Θ.

(d) Calculate $E[\Theta]$ for the case $\lambda = \mu$.

You can use without verification the following formulae

$$\int_0^\infty r e^{-r\alpha} dr = \alpha^{-2}, \qquad \alpha > 0,$$

$$\int \frac{x\ dx}{1 + \sin x} = -x\ \tan\left(\frac{\pi}{2} - \frac{x}{2}\right) + 2 \ln \left|\cos\left(\frac{\pi}{4} - \frac{x}{2}\right)\right|.$$

5. Suppose X is a discrete random variable with Probability Mass Function

$$P(X = k) = p(k), \quad k = 0, 1, 2, \ldots$$

(a) State the definition of the Moment Generating Function of X.

(b) Consider a discrete random variable Y with Probability Mass Function

$$p(k) = \left(\frac{1}{2}\right)^{k+1}, \quad k = 0, 1, 2, \ldots$$

Find $M_Y(t)$, the Moment Generating Function of Y.
Indicate the range of values of t for which $M_Y(t)$ exists.

(c) Using the Moment Generating Function find $E[Y]$ and $E[Y^2]$.

(d) Calculate $Var\ Y$.

6. The random variables $X_1, \ldots, X_n$ are independent and identically distributed, with the Probability Mass Function $P(X = -1) = P(X = +3) = 0.5$.

(a) Calculate $\mu = E[X]$ and $\sigma^2 = Var\ X$.

(b) For the case $n = 200$ use the Central Limit Theorem to approximate the probability

$$P\{\ln(\bar{X}) > 0\},$$

where

$$\bar{X} = \frac{1}{200}\sum_{i=1}^{200} X_i$$

is the sample mean and ln denotes the natural logarithm (i.e. logarithm with base e).

(c) Find the minimum n for which

$$P\{\sum_{i=1}^{n} X_i > 190\} > 0.99.$$

7. (a) State the definition and the main properties of the gamma-function $\Gamma(t)$.

(b) State the definition of a chi-square distribution in terms of standard normal random variables.

(c) Suppose U and V are independent chi-square random variables with m and n degrees of freedom, respectively. Show that the random variable W defined by

$$W = \frac{U/m}{V/n}$$

has density

$$f_W(w) = \frac{\Gamma(\frac{m+n}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})}\left(\frac{m}{n}\right)^{m/2} w^{\frac{m}{2}-1}\left(1 + \frac{m}{n}w\right)^{-\frac{m+n}{2}}, \quad w > 0.$$

Hint: Recall that the density of a chi-square distribution with n degrees of freedom is

$$f_V(v) = \frac{(1/2)^{n/2}}{\Gamma(\frac{n}{2})} v^{\frac{n}{2}-1} e^{-v/2}.$$

Recall also that if A and B are independent random variables, then the density of the quotient $C = B/A$ is

$$f_C(c) = \int |a| f_A(a) f_B(ca) da.$$

Finally, recall that the gamma density is

$$f_{\alpha,\lambda}(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \ x > 0, \qquad \alpha, \lambda > 0.$$

[NORMAL DISTRIBUTION TABLE]