**MATH762 May 2006**
**time: 2.5 hours**

### Instructions to candidates

You may attempt all questions

The best five answers will be taken into account.

The marks noted indicate the relative weight of the questions.

Note that all interest rates quoted are per annum; and unless stated otherwise, assume that all interest rates and dividend rates are continuously compounded. Tables of the cumulative normal distribution ($N(x) \equiv \Phi(x)$) are appended.

**1.** (a) Draw a graph of the value at the expiry date versus share price $S$ at that date for the following portfolios:

(i) A share and a shorted (i.e. written) put option with strike price 100p, both for the same underlying share.

(ii) A put option with strike price 200p and a call option with strike price 300p, both for the same underlying share.

(b) Explain carefully using arbitrage arguments why the price for a future contract on an asset paying no dividend is given by

$$F = Se^{rt},$$

where $S$ is the asset price when the contract is made, $t$ is the time between when the contract is made until it is fulfilled and the risk-free rate of interest is $r$.

Indicate briefly how the result is altered if (a) the asset pays a single dividend $D$ during the contract period, and (b) if the asset pays a continuously compounded dividend with dividend rate $d$.

If on a certain date the FTSE100 index is at 6000, calculate (to 2 decimal places) the price of a futures contract to deliver this index to a customer 8 months in the future (with payment of $F$ then). You may assume that the continuously compounded dividend paid on the index is 3% and the risk free rate of interest is 4.5%.

(c) On a certain date shares in a certain company are priced at 700p and there are in total five million shares. What is the market capitalisation of the company in pounds?

The company makes a rights issue of three share for every existing share, priced at 300p. What will the share price be after the rights issue? Explain briefly the consequences for a shareholder of (i) taking up the rights issue (ii) not taking up the rights issue.

[20 marks]

**2.** (a) A random variable $X$ has a probability density function given by

$$f(x) = ae^{-x} \quad \text{when} \quad x \geq 0$$

and

$$f(x) = 0 \quad \text{when} \quad x < 0,$$

where $a$ is a constant.

Calculate $a$, $E(X)$ and $E(X^2)$.

(b) For consecutive months, the price S of a share was as follows (in pence):

600 615 592 598 623 686 662 676 690

Assuming this share follows a log-normal random walk, evaluate the mean growth rate $\mu$ (of $\ln S$) and volatility $\sigma$ from these data. Express your results to 3 significant figures using time units of years.

With these same assumptions, write down and evaluate (using tables for $N(x)$; you need not use interpolation) an integral which gives the probability that the value of a share will be less than 750 three months after it is 690. Give your result to 2 significant figures.

[20 marks]

**3.** Consider a model of share price behaviour where the share price $S$ may increase or decrease by a factor of 5/4 or 4/5 in each 1 month period. Use this model to value (to the nearest penny) a put option with strike price 6000p and expiry in two months when the current share price is 5600p. You may assume that the risk-free rate of interest is 5%.

[20 marks]

**4.** (a) Stating your assumptions, derive the relationship between the share price $S$, the call option price $C$ and the put option price $P$ when the options have a common expiry date at time $T$ in the future and a common strike price $E$. You may neglect dividends and assume that the risk-free rate of interest is $r$.

(b) The explicit solution of the Black-Scholes equation for the price of a call option $C(S(t), t)$ at time $t$ with expiry at time $T_E$, strike price $E$ and current share price $S$ with risk free rate of interest $r$ and volatility $\sigma$ is:

$$C(S,t) = SN(d_1) - Ee^{-rT}N(d_2)$$

with $T = T_E - t$, $\sigma\sqrt{T}d_1 = \ln(S/E) + (r + \sigma^2/2)T$ and $\sigma\sqrt{T}d_2 = \ln(S/E) + (r - \sigma^2/2)T$. What assumption concerning dividends on the share is required to obtain the above solution?

Explain the significance of the function $N(x)$, and give the values of $N(\infty)$ and $N(-\infty)$.

$C(S(t), t)$ satisfies definite boundary conditions if (i) $t = T_E$, (ii) $S = 0$, (iii) $S \to \infty$. Write down these boundary conditions and explain carefully (using arbitrage arguments where appropriate) why they must be satisfied. Verify that the explicit solution given above satisfies all the boundary conditions.

[20 marks]

**5.** A “cash or nothing” (also called “all or nothing”) call option $V$ (on a share with price $S$) has a value at expiry ($t = T_E$) of $\mathcal{B}$ if $S > E$ and zero if $S \leq E$ where $\mathcal{B}$, $E$ are constants.

Express the expiry value of $V$ in terms of the Heaviside function.

(i) Consider a model of share price behaviour where a share valued at 130p may increase to 160p over 1 month with probability $p$ and may decrease in value to 100p over 1 month with probability $(1-p)$. Use this model, neglecting the cost of borrowing money, to value a “cash or nothing” call option with $E = 140p$, $\mathcal{B} = 100p$, and expiry date in 1 month. Explain the assumptions you are making.

(ii) The solution of the Black-Scholes equation for a “cash or nothing” call option $V$ is (for $t < T_E$):

$$V = \mathcal{B}e^{-rT}N(d_2)$$

where $T = T_E - t$ and $\sigma\sqrt{T}d_2 = \ln(S/E) + (r - \sigma^2/2)T$. Here $r$ is the risk free rate of interest and $\sigma$ is the volatility.

Using arbitrage arguments, derive the analogous solution for a “cash or nothing” put option $V'$ which has a value at expiry ($t = T_E$) of $\mathcal{B}$ if $S \leq E$ and zero if $S > E$.

Calculate the prices of such options $V$, $V'$ with expiry in 6 months, if the current share price $S = 600p$, $E = 550p$ and $\mathcal{B} = 100p$. Assume that the risk free rate of interest is 5% and the volatility is 0.4 (in time units of years), and use tables of $N(x)$ with interpolation if necessary. Give your answer to 4 significant figures.

[20 marks]

**6.** Consider a continuous random walk in the share price

$$dS = \sigma S dX + (\mu - d)S dt.$$

Explain the meaning of the symbols in this expression.

Show that a smooth function $V(S,t)$ satisfies the equation

$$dV = \sigma S \frac{\partial V}{\partial S} dX + \left[ (\mu - d)S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right] dt$$

explaining carefully any assumptions you make. Using this result, present the derivation of the Black-Scholes differential equation satisfied by an option $V(S,t)$ for the case when the underlying asset $S$ pays a continuously compounded dividend with dividend rate $d$ and the risk-free rate of interest is $r$.

For such an option $V(S,t)$, show that if we define $W = e^{d(T_E - t)}V$, where $T_E$ is the option expiry time, then $W$ satisfies an equation identical to the Black-Scholes equation *without* dividends but with $r$ replaced by $r' = r - d$.

[20 marks]