MATH762 May 2005

Instructions to candidates

You may attempt all questions

The best five answers will be taken into account.

The marks noted indicate the relative weight of the questions.

Note that all interest rates quoted are per annum and that tables of the cumulative normal distribution ($N(x) \equiv \Phi(x)$) are appended.

1. (a) Draw a graph of the value at the expiry date versus share price S at that date for the following portfolios:

(i) A share and a call option (on the same underlying share) with strike price 400p.

(ii) A put option with strike price 400p and a shorted (ie written) put option with strike price 200p, both for the same underlying share.

(b) On a certain date when the FTSE100 index is at 5000, a broker writes a futures contract at price F to deliver this index to a customer 9 months in the future (with payment of F then). Describe the risk-free strategy available to the broker to provide this futures contract and hence obtain the price F (to 2 d.p.) of the futures contract. You may assume that the dividend paid on the index is 3% and the risk free rate of interest is 5%.

If the futures contract is available at a price more than F, describe how to make a risk-free profit.

(c) Stating your assumptions, derive the relationship between the share price S, the call option price C and the put option price P when the options have a common expiry date at time t in the future and a common strike price E. You may neglect dividends and assume that the risk-free rate of interest is r.

A put option with expiry in 6 months and strike price 600p is quoted at 130p. The share price is quoted at 652p. Assuming the risk free rate of interest is 5% and neglecting any dividend payments, value a call option (to nearest penny) with the same expiry date and strike price. [20 marks]

2. (a) Consider a model of share price behaviour where a share valued at 120p may increase to 140p over 1 month with probability p and may decrease in value to 100p over 1 month with probability $(1-p)$. Use this model, neglecting the cost of borrowing money, to value a call option with strike price 110p and expiry date in 1 month. Explain the assumptions you are making.

Also value a put option with the same strike price and expiry date.

(b) The explicit solution of the Black-Scholes equation for the price of a put option at time t with expiry at time T_E, strike price E and current share price S with risk free rate of interest r and volatility σ is given by:

$$P(S,t) = Ee^{-rT}N(-d_2) - SN(-d_1)$$

with $T = T_E - t$, $\sigma\sqrt{T}d_1 = \ln(S/E) + (r + \sigma^2/2)T$ and $\sigma\sqrt{T}d_2 = \ln(S/E) + (r - \sigma^2/2)T$

Use this solution to evaluate (using tables for N with interpolation where necessary) the price of a put option (to 1 decimal place) with expiry in 4 months, strike price 600p and current share price 500p. You may assume that the risk free rate of interest is 5% and the volatility is 0.35 (in time units of years).

How many shares are to be bought (or sold short) initially as a hedging strategy to be used by a writer of 5000 such put options?

[20 marks]

3. (a) If a random variable S is distributed according to a log-normal distribution (with $\ln S$ having mean $\bar{l} = \ln 600$ and variance $\sigma^2 = 0.08$), find the probability (to 3 significant figures) that $S > 400$. You may use tables of N (with interpolation where necessary).

(b) For consecutive weeks, the price S of a share was as follows (in pence):

2170 2214 2508 2458 2610 2784 2830 2632 2892 3026

Assuming this share follows a log-normal random walk, evaluate the mean growth rate μ (of $\ln S$) and volatility σ from these data. Express your results to 3 significant figures using time units of years.

With these same assumptions, derive an expression (as an integral which you need not evaluate) for the expected value of a put option with strike price 2900p and expiry in 3 months when the current share price is 3026p.

[20 marks]

4. Consider a model of share price behaviour where the share price S may increase or decrease by a factor of $9/8$ or $8/9$ (with probabilities p, $(1-p)$ respectively) in each 1 month period. Use this model to value (to 1 d.p.) a call option with strike price 5500p and expiry in two months when the current share price is 5184p. You may neglect the cost of borrowing.

Describe the hedging strategy to be used by a writer of a call option for the above model of share pricing.

[20 marks]

5. The Fourier Transform of a function $f(x)$ defined on the interval $-\infty < x < \infty$ is

$$F\{f(x);k\} = \overline{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx,$$

and satisfies the equation

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{f}(k)e^{ikx}\,dk.$$

Show that if $f(x)$ satisfies $f, f_x \to 0$ as $x \to \pm\infty$, then

$$F\{\frac{df}{dx};k\} = ik\overline{f}(k),$$

and

$$F\{\frac{d^2f}{dx^2};k\} = (ik)^2\overline{f}(k).$$

Now assume that $u(x,t)$ satisfies the diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

for $t \geq 0$, with boundary conditions as follows:

$$u(x,0) = u_0(x), \quad u, u_x \to 0 \text{ as } x \to \pm\infty.$$

Show that the Fourier transform of u with respect to x, $\overline{u}(k,t)$, satisfies the equation

$$\frac{\partial \overline{u}}{\partial t} = -k^2\overline{u}.$$

Hence or otherwise show that $u(x,t)$ is given by:

$$u(x,t) = \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} u_0(x')e^{-\frac{(x-x')^2}{4t}}\,dx',$$

and find $u(x,t)$ for the case $u_0(x) = A\delta(x)$, where A is a constant, and $\delta(x)$ is the Dirac δ-function. Verify explicitly that the resulting expression satisfies the diffusion equation.

[20 marks]

6. Consider a continuous random walk in the share price

$$dS = S\sigma dX + S\mu dt.$$

Explain the meaning of the symbols in this expression.

Show that a smooth function $V(S,t)$ satisfies the equation

$$dV = \sigma S \frac{\partial V}{\partial S} dX + (\mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}) dt$$

explaining carefully any assumptions you make.

Ignoring dividends and assuming a risk-free rate of interest r, present the derivation of the Black-Scholes differential equation satisfied by an option $V(S,t)$.

The explicit solution of the Black-Scholes equation for the price of a call option at time t with expiry at time T_E, strike price E and current share price S with risk free rate of interest r and volatility σ is given by:

$$C(S,t) = SN(d_1) - Ee^{-rT}N(d_2),$$

with T, d_1, and d_2 as defined in Question 2. Prove that

$$\Delta = \frac{\partial C}{\partial S} = N(d_1).$$

[20 marks]