PAPER CODE NO.
MATH762

SUMMER 2004 EXAMINATIONS

Bachelor of Arts : Year 4
Bachelor of Science : Year 3
Bachelor of Science : Year 4

INTRODUCTION TO FINANCIAL MATHEMATICS

TIME ALLOWED : Two Hours and a Half

INSTRUCTIONS TO CANDIDATES

You may attempt all questions
The best five answers will be taken into account.
The marks noted indicate the relative weight of the questions.
Note that all interest rates quoted are per annum and that tables of the cumulative normal distribution ($N(x) \equiv \Phi(x)$) are appended.

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1. (a) Draw a graph of the value at the expiry date versus share price S at that date for each of the following portfolios:

   (i) A share and a put option with exercise price 1000p.

   (ii) A call option with exercise price 1000p and a shorted (ie written) put option with exercise price 600p.

   (b) On a certain date the FTSE100 index is at 8000 and the price of a futures contract to buy this index 6 months in the future is 8150. You may assume that the dividend paid on the index is 2% and the risk free rate of interest is 5%. Give arbitrage arguments to show whether it is theoretically possible to make a risk free profit - and if so how.

   (c) Stating your assumptions, derive the relationship between the share price S, call option price C and put option price P when the options have a common expiry date at time T in the future and a common strike (ie exercise) price E. [You may neglect dividends and assume the risk-free interest rate is r.]

   A call option with expiry in 6 months and strike price 600p is quoted at 130p. The share price is quoted at 610p. Assuming the risk-free rate of interest is 5% and neglecting any dividend payments, value a put option with the same expiry date and strike price.

   (d) If a call option with expiry in 6 months and strike price 600p is quoted at 100p, and if the share price is quoted at 700p, show that it is theoretically possible to make a risk-free profit. [Hint: consider a portfolio made up of shares and call options.] You may assume that the risk-free interest rate is 5% and neglect any dividend payments.

   Explain briefly the transactions required in order to make a risk-free profit. What is the minimum risk-free profit?

[20 marks]

2. (a) Consider a model of share price behaviour where a share valued at 500p may increase in value to 600p or may decrease in value to 400p over 4 months. Use this model, neglecting the cost of borrowing money, to value a call option with exercise price 550p and expiry date in 4 months. Explain clearly the method you are using.

Also value the same call option with the same strike price and expiry date taking account of the cost of borrowing at the risk-free rate of 5%.

(b) In this part of the question you may assume that the explicit solution of the Black-Scholes equation for the price of a call option at time t with expiry at time T_E, strike price E, current share price S, risk free rate of interest r and volatility σ is:

$$C(S,t) = SN(d_1) - Ee^{-rT}N(d_2)$$

with $T = T_E - t$, $\sigma\sqrt{T}d_1 = \ln(S/E) + (r + \sigma^2/2)T$ and $\sigma\sqrt{T}d_2 = \ln(S/E) + (r - \sigma^2/2)T$

Use this explicit solution of the Black-Scholes equation to evaluate (using tables for N with interpolation where necessary) the price of a call option (to 1 decimal place) with expiry in 4 months, strike price 400p and current share price 450p. You may assume that the risk free rate of interest is 3% and the volatility is 0.2 (in time units of years).

(c) Using put-call parity, show that the value of a put option may be expressed as

$$P(S,t) = Ee^{-rT}N(-d_2) - SN(-d_1)\ .$$

[20 marks]

3. (a) For consecutive months the price S of a certain share was as follows (in pence):

   540 550 555 512 470 520 540 576 590 580

   Assuming this share follows a log-normal random walk, evaluate the mean growth rate (of $\ln S$) and volatility from these data. Express your results to 3 significant figures using time units of years.

   (b) Assuming a random walk in the log of a share price with no growth term (in $\ln S$) and with value 0.2 for the volatility in time units of years, estimate the probability (to precision of 0.1%) that this share will be worth more than 610p three months later when its current price is 580p. [Tables of N may be used]

   (c) Evaluate the probability for the situation as in question 3(b) but with the change that the growth rate is to be taken as that evaluated in question 3(a) rather than as zero.

[20 marks]

4. Consider a model of share price behaviour where the share price S may increase by a factor of 4/3 or decrease by a factor of 3/4 in each two month period. Use this model to value (to 1 decimal place) a call option with strike price 160p and expiry in four months when the current share price is 144p. You may neglect the cost of borrowing.

   Describe the hedging strategy to be used by the writer of a call option for the above model of share pricing.

   For the case when $p = 1/2$, consider the above model as a discrete random walk using as a variable $X = \ln S$. Obtain an expression for the variance of X after n steps assuming $X = x_o = \ln 144$ initially. Hence estimate the volatility in this model.

[20 marks]

5. Consider a continuous random walk in the share price

$$dS = S\sigma dX + S\mu dt$$

where you should explain the meaning of the symbols in this expression.

Show that the change in the value $V(S,t)$ of an option is given by

$$dV = \sigma S \frac{\partial V}{\partial S} dX + (\mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}) dt$$

Explain the assumptions used in the derivation of the Black-Scholes equation.

Assuming the risk-free rate of interest is r and ignoring dividends, present the derivation of the Black-Scholes differential equation satisfied by $V(S,t)$.

Consider the boundary conditions on $V(S,t)$ appropriate for a call option with exercise price E and expiry time T. Give expressions with explanation for $V(S,T)$, $V(0,t)$ and the limit as $S \to \infty$ of $V(S,t)$.

[**20 marks**]

6. (a) If S is a random variable satisfying the stochastic differential equation

$$dS = S\sigma dX + S\mu dt$$

where the expectation values are given by $E(dX) = 0$ and $E(dX^2) = dt$, derive Ito's lemma (stating carefully any assumptions you make), and use it to find the corresponding equation for the random variable $g(S) = S^3$.

(b) The standard diffusion equation is

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$$

Verify that $u_\delta(x, \tau) = c\tau^{-1/2}e^{-x^2/4\tau}$ with constant c is a solution to the standard diffusion equation.

Find c by normalizing $u_\delta(x, \tau)$.

Obtain an expression (in terms of integrals which you need not evaluate) for the solution for $\tau > 0$ given the following initial condition

$$u(x, 0) = max(2x, 0) - x \; .$$

(c) Consider the partial differential equation, where a and b are constant,

$$\frac{\partial v}{\partial t} = a + b\frac{\partial^2 v}{\partial x^2}$$

Make substitutions to reduce this to the standard diffusion equation for $u(x, \tau)$.

[20 marks]