MATH762 2003

Instructions to candidates

You may attempt all questions

The best five answers will be taken into account.

The marks noted indicate the relative weight of the questions.

Note that all interest rates quoted are per annum and that tables of the cumulative normal distribution ($N(x) \equiv \Phi(x)$) are appended.

1. (a) Draw a graph of the value at the expiry date versus share price S at that date for the following portfolios:

(i) A share sold short and a call option (on the same underlying share) with strike price 400p.

(ii) A put option with strike price 300p and a shorted (ie sold short or written) call option with strike price 500p, both for the same underlying share. [6 marks]

(b) On a certain date when the FTSE100 index is at 4000, a broker writes a futures contract at price F to deliver this index to a customer 6 months in the future (with payment of F then). Describe the risk-free strategy available to the broker to provide this futures contract and hence obtain the price F (to 2 d.p.) of the futures contract. You may assume that the dividend paid on the index is 1% and the risk free rate of interest is 3%.

If the futures contract is available at a price less than F, describe how to make a risk-free profit. [7 marks]

(c) Stating your assumptions, derive the relationship between the share price S, the call option price C and the put option price P when the options have a common expiry date at time t in the future and a common strike price E. You may neglect dividends and assume that the risk-free rate of interest is r.

A put option with expiry in 3 months and strike price 500p is quoted at 126p. The share price is quoted at 550p. Assuming the risk free rate of interest is 4% and neglecting any dividend payments, value a call option (to nearest penny) with the same expiry date and strike price. [7 marks]

2. (a) Consider a model of share price behaviour where a share valued at 600p may increase to 700p over 1 month with probability p and may decrease in value to 500p over 1 month with probability $(1-p)$. Use this model, neglecting the cost of borrowing money, to value a call option with strike price 550p and expiry date in 1 month. Explain the assumptions you are making.

Also value a put option with the same strike price and expiry date.

[10 marks]

(b) In this question you may assume the explicit solution of the Black Scholes equation for the price of a call option at time t with expiry at time T_E, strike price E and current share price S with risk free rate of interest r and volatility σ is:

$$C(S,t) = SN(d_1) - Ee^{-rT}N(d_2)$$

with $T = T_E - t$, $\sigma\sqrt{T}d_1 = \ln(S/E) + (r + \sigma^2/2)T$ and $\sigma\sqrt{T}d_2 = \ln(S/E) + (r - \sigma^2/2)T$

Use this explicit solution of the Black Scholes equation to evaluate (using tables for N with interpolation where necessary) the price of a call option (to 1 decimal place) with expiry in 4 months, strike price 380p and current share price 300p. You may assume that the risk free rate of interest is 4% and the volatility is 0.35 (in time units of years).

How many shares are to be bought (or sold short) initially as a hedging strategy to be used by a writer of 1000 such call options? [10 marks]

3. (a) If a random variable S is distributed according to a log-normal distribution (with $\ln S$ having mean $\bar{l} = \ln 300$ and variance $\sigma^2 = 0.09$), find the probability (to 3 significant figures) that $S < 200$. You may use tables of N, with interpolation where necessary. [6 marks]

(b) For consecutive months, the price S of a share was as follows (in pence):

451 473 480 460 432 412 413 403 423 402 398 404

Assuming this share follows a log-normal random walk, evaluate the mean growth rate μ (of $\ln S$) and volatility σ from this data. Express your results to 3 significant figures using time units of years. [8 marks]

With these same assumptions, derive an expression (as an integral which you need not evaluate) for the expected value of a call option with strike price 400p and expiry in 3 months when the current share price is 404p. [6 marks]

4. Consider a model of share price behaviour where the share price S may increase or decrease by a factor of 6/5 or 5/6 (with probabilities p, $(1-p)$ respectively) in each 1 month period. Use this model to value (to 1 d.p.) a call option with strike price 840p and expiry in two months when the current share price is 900p. You may neglect the cost of borrowing. [10 marks]

Describe the hedging strategy to be used by the writer of such a call option for the above model of share pricing. [5 marks]

For the case when $p = 1/2$, consider this discrete random walk using as a variable $X = \ln S$. Obtain an expression for the variance of X after n random steps assuming $X = x_0 = \ln 900$ initially and hence find the variance after one year. Also estimate the volatility in this model (to 3 significant figures).

[5 marks]

5. Consider a continuous random walk in the share price

$$dS = S\sigma dX + S\mu dt$$

where you should explain the meaning of the symbols in this expression.

Explain the assumptions used in the derivation of the Black Scholes equation for the value $V(S,t)$ of an option.

Explain why the assumptions used do not apply to a futures contract.

Neglecting the cost of borrowing and ignoring dividends, present the derivation of the Black Scholes differential equation satisfied by $V(S,t)$.

Assuming dividends are paid continuously at rate D_0, derive the modification to the Black Scholes equation (still assuming that the cost of borrowing can be neglected).

[15 marks]

Consider the boundary conditions on $V(S,t)$ appropriate for a call option with strike price E and expiry time T. Give expressions with explanation for $V(S,T)$, $V(0,t)$ and the limit as $S \to \infty$ of $V(S,t)$. [5 marks]

6. (a) If S is a random variable satisfying the stochastic differential equation

$$dS = S\sigma dX + S\mu dt$$

where the expectation values are given by $E(dX) = 0$ and $E(dX^2) = dt$, use Ito's lemma to find the corresponding equation for the random variable $g(S) = \ln(S^2)$. [6 marks]

(b) The standard diffusion equation is

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$$

Verify that $u(x, \tau) = c\tau^{-1/2} e^{-x^2/(4\tau)}$ with constant c is a solution to the standard diffusion equation. [5 marks]

(c) Consider the partial differential equation

$$\frac{\partial v}{\partial \tau} = S^2 \frac{\partial^2 v}{\partial S^2}$$

Using the substitutions $S = e^x$ and $v(S, \tau) = u(x, \tau)e^{x/2-\tau/4}$, reduce this to the standard diffusion equation for u and hence give one non-trivial solution for v in terms of x and τ. [9 marks]