1. (a) A manufacturing firm intends to start making three new products, called 1, 2 and 3. The available capacity on the machines involved in production of the products is as follows:

Machine type	Available time (hours/week)
Milling machine	400
Lathe	200
Grinder	150

The number of machine hours required for each product is as follows:

Machine type	Product 1	Product 2	Product 3
Milling machine	9	2	4
Lathe	5	7	0
Grinder	3	1	2

The sales potential for Products 1 and 2 is effectively unlimited, but Product 3 has a maximum sales potential of 30 per week. The unit profits are £50, £25 and £30, respectively, for Products 1, 2 and 3.

The company wishes to maximise its weekly profit. Formulate this problem as a linear program. (Do **not** go on to solve it.) [7 marks]

Formulate the dual problem. [3 marks]

(b) A television manufacturing company makes two kinds of television sets, namely 27-inch and 20-inch. Market research shows that at most 30 of the 27-inch and 20 of the 20-inch sets can be sold per month. The company's workforce can provide a maximum of 400 work hours per month. Each 27-inch set requires 20 work hours, and each 20-inch requires 10 work hours. A 27-inch set can be sold to give a profit of £80, while every 20-inch set brings a profit of £60.

The company wishes to decide how many sets of each kind to produce each month. Formulate this problem as a linear program, including units in your answer.

Sketch the feasible region of the problem, and find the optimal solution.

[10 marks]

2. (a) Use the primal simplex method to solve the linear program:
maximise $x_0 = 2x_1 - x_2 + 2x_3$
subject to

$$\begin{aligned} 4x_1 + x_2 &\leq 8 \\ -x_2 + 2x_3 &\leq 8 \\ -4x_2 + 4x_3 &\leq 8 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Does this problem have any other solutions? [10 marks]

(b) Use the dual simplex method to solve the following linear program. (Solution by any other method will not receive credit.)
maximise $x_0 = -4x_1 - 3x_2 - 3x_3$
subject to

$$\begin{aligned} x_1 + x_2 - x_3 &\geq 10 \\ 2x_1 - x_2 + x_3 &\geq 16 \\ 2x_1 + x_2 - 2x_3 &\geq 14 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Does this problem have any other solutions? [10 marks]

3. Consider the following linear program:
minimise $y_0 = 48y_1 + 20y_2 + 13y_4$
subject to

$$\begin{aligned} 3y_1 + 2y_2 - 3y_3 + y_4 &\geq 3 \\ 4y_1 + y_2 + y_3 + y_4 &\geq 2 \\ y_1, y_2, y_3, y_4 &\geq 0 \end{aligned}$$

(a) Formulate the dual to it using variables x_1, x_2 and solve it graphically. Are there any redundant constraints? Which constraints are binding? [10 marks]

(b) Write down all the complementary slackness conditions and use them to find the optimal solution $y_1^*, y_2^*, \ldots$ to the original linear program. [10 marks]

4. (a) Within the context of bicriterion problems, say what it means for a point $\mathbf{y}$ to be inferior to a point $\mathbf{x}$ in a minimisation problem with objective functions Z_1 and Z_2. [1 marks]

(b) Consider the bicriterion problem

minimise $\{Z_1, Z_2\}$

subject to

$$\begin{aligned} x + y &\leq 5 \\ -x + y &\leq 3 \\ x - y &\leq 3 \\ 2x + y &\geq 3 \\ x + 2y &\geq 3 \\ x, y &\geq 0. \end{aligned}$$

(i) Sketch the feasible region for this problem. [4 marks]

(ii) If $Z_1 = 3x + y$ and $Z_2 = x + 3y$, state the values of x, y, Z_1 and Z_2 at each vertex of the feasible region. Identify which of these vertices are inferior solutions and hence find the Non-Inferior Set (NIS). What constraints can be omitted so that NIS remains the same? [7 marks]

(iii) Let $Z(w) = (1 - w)Z_1 + wZ_2$. Determine, as a function of w, where $0 \leq w \leq 1$, the set of points of the feasible region for which $Z(w)$ is minimised. [4 marks]

(iv) Suppose the goals for Z_1 and Z_2 are $G_1 = 2$ and $G_2 = 0$, respectively. Formulate a goal program in which the penalties for exceeding goals G_1 and G_2 are 3 and 5 per unit, respectively, and where there is no penalty for undershooting. [4 marks]

5. (a) Consider a single-item static continuous review model in which the set-up cost is K per order, the demand rate is D units per unit time and the holding cost is h per unit per unit time. If the order size is y units, then the Total Cost per Unit time is given by

$$TCU(y) \quad = \quad \frac{KD}{y} + \frac{hy}{2}.$$

(i) Given that the Economic Order Quantity y^* is given by $y^* = \sqrt{2KD/h}$, find an expression for $TCU\left(y^*\right)$. [2 marks]

(ii) Calculate the values of y^* and $TCU\left(y^*\right)$ if $K = £50$ per order, $D = 8$ units per day and $h = £0.10$ per unit per day. [1 marks]

(iii) Show that, in general,

$$\frac{TCU(y)}{TCU\left(y^*\right)} \quad = \quad \frac{1}{2}\left(\left(\frac{y}{y^*}\right) + \left(\frac{y^*}{y}\right)\right)$$

[3 marks]

(iv) Given the numerical values of K, D, h above, for what range of y values will $TCU(y)$ be within 15% of its minimal value $TCU\left(y^*\right)$? [3 marks]

(b) Consider now a single-item static continuous review model in which shortages are permitted. Suppose the set-up cost is K per order, the demand rate is D units per unit time, the holding cost is h per unit per unit time, and the shortage cost is p per unit per unit time. It is known that in such model the Total Cost per Unit time is given by

$$TCU(y, w) = \frac{KD}{y} + \frac{h(y-w)^2}{2y} + \frac{pw^2}{2y},$$

where the order size is y units and shortages of up to w units are permitted to occur.

(i) Sketch the typical graph of stock level against time. [3 marks]

(ii) By differentiating $TCU(y, w)$ with respect to w and setting the derivative equal to zero, show that the values of y and w which minimise cost satisfy

$$\frac{w}{y} = \frac{h}{h+p}.$$

[3 marks]

(iii) Hence by differentiating $TCU(y, w)$ with respect to y, find expressions in terms of $h,\ p,\ K$ and D for the values of y and w which minimise cost. [5 marks]

6. A company produces K units of a product monthly. It can sell any amount of $x \leq K$ units to an existing customer for a fixed price α £/unit. The remaining part of y units should be sold on the market for a flexible price: in order to be able to sell y units the price should be

$$p(y) = \beta + \frac{1}{\sqrt{y}} \qquad \text{£/unit.}$$

Assume that $\alpha > \beta$. The company has to sell all K units this month, the goal is to maximise total income.

(a) Formulate this problem as a nonlinear program clearly indicating all the constraints. [5 marks]

In parts (b) and (c), ignore the trivial non-negativity constraints.

(b) Write down the Lagrange function and formulate the necessary conditions of optimality. [7 marks]

(c) Find the optimal values of x and y, in terms of α, β and K, providing the maximal total income. [6 marks]

(d) For what values of α, β, K the solution obtained makes sense? [2 marks]

7. (a) A company has 3 plants producing a product to be shipped to 4 distribution centres. Plants 1, 2 and 3 produce 14, 15 and 11 shipments per month, respectively. Each distribution centre needs to receive 10 shipments per month. The distance from each plant to each distribution centre is given in miles as follows.

		Distribution centre			
		1	2	3	4
	1	400	700	1100	1400
Plant	2	600	1000	600	1200
	3	800	900	800	1300

The freight cost for each shipment is £100 plus £0.50 per mile. It is required to know how much should be shipped to each plant from each distribution centre so as to minimise the total shipping cost. Formulate this problem as a transportation problem and use the North West Corner Rule to find an initial basic feasible solution. [6 marks]

(b) (i) Solve the following transportation problem, starting from the initial basic solution given:

	P	Q	R	S
A	$\boxed{10}$ 6	$\boxed{0}$ 10	$\boxed{20}$	$\boxed{11}$
B	$\boxed{12}$	$\boxed{7}$ 6	$\boxed{9}$ 15	$\boxed{20}$ 5
C	$\boxed{0}$	$\boxed{14}$	$\boxed{16}$	$\boxed{18}$ 5

[10 marks]

(ii)Suppose now that the available supply at A is increased by 5 units. Explain how this new feature may be modelled as a balanced transportation problem. (NB you should **not** solve this new problem.) [4 marks]