1. (a) A company produces a breakfast cereal from a mixture of oats and wheat. The availability of oats is limited to 15000 kg per week and that of wheat to 10000 kg per week. The market for the cereal is limited to 20000 kg per week. Each kg of oats used yields a profit of 50 pence and each kg of wheat used yields a profit of 30 pence. Let x_O and x_W denote, respectively, the amounts of oats and wheat used (in kg per week). Formulate a linear program (including units in your answer) to solve the problem of determining x_O and x_W so as to maximise the company's profit. Do **not** go on to solve this linear program.

[**Note** that you may assume that 1 kg of ingredients yields 1 kg of cereal; that is, there is no 'weight loss'.]

[6 marks]

(b) A fruit processing company has contracted to buy 100,000 kg of apples from which it will produce apple juice and apple preserve. A bottle of juice requires 1 kg of apples and sells for £1.00 whereas a jar of preserve requires 0.5 kg of apples and sells for £2.00. Production cost is 40p per kg of apples whether juice or preserve is being made. However, the production of each bottle of juice also yields an amount of apple pulp worth 10p. The company can sell up to 50,000 jars of preserve and is contracted to produce 60,000 bottles of juice, but otherwise its production policy is unconstrained. Formulate a linear program (including units in your answer) to determine the numbers of bottles of juice and jars of preserve that should be produced in order to maximise profit. Do **not** go on to solve the linear program.

[10 marks]

Suppose now that the company considers buying an extra amount, y kg, of apples at 45 pence per kilogram. Modify your formulation to determine the value of y as well as the numbers of bottles of juice and jars of preserve that should be produced in order to maximise profit.

[4 marks]

2. (a) Sketch the feasible region for the linear program

maximise $z = x + 2y$

subject to

$$
\begin{aligned}
x + y &\geq 1 \\
2x + 3y &\leq 6 \\
4x + 3y &\leq 12 \\
x, y &\geq 0
\end{aligned}
$$

Determine the optimal solution and its value.

State which constraints are binding at optimality and which are non-binding.

Which (if any) constraints are redundant?

[7 marks]

(b) Use the simplex method to solve

maximise $x_0 = 8x_1 - 2x_2 - 5x_3$

subject to

$$
\begin{aligned}
-4x_1 + 4x_2 + x_3 &\leq 4 \\
4x_1 \quad + \quad 2x_3 &\leq 44 \\
2x_1 - x_2 - x_3 &\leq 10 \\
x_1, x_2, x_3 &\geq 0.
\end{aligned}
$$

Check that your solution satisfies the constraints. What are the basic variables in your optimal solution? State whether there is an alternative optimal basis, and if so then write one down.

[9 marks]

State, giving reasons, whether your optimal basis is *primal* and/or *dual* degenerate.

[4 marks]

3. (a) Sketch the feasible region of the problem

$$\begin{aligned} &\text{maximise } z = ax + 2y \\ &\text{subject to} \\ &\qquad x + y \le d \\ &\qquad -x + y \le 1 \\ &\qquad x + ky \le 3 \\ &\qquad x, y \ge 0. \end{aligned}$$

when d and k take the values 4 and 0 respectively. Identify the optimal solution for $a = 1$. Answer the following questions in each of which just one of d, k and a is varied from the values given above.

(i) To what value must d increase before $x + y \le d$ becomes redundant?

(ii) Find the range of d for which the constraint $x + y \le d$ is binding at optimality.

(iii) Find the range of a for which the optimal solution remains optimal.

(iv) By how much can k increase without the optimal solution being affected?

[10 marks]

(b) Solve the linear program

$$\begin{aligned} &\text{maximise } x_0 = -9x_1 - 3x_2 - 2x_3 \\ &\text{subject to} \\ &\qquad x_1 + x_2 + 4x_3 \le 11 \\ &\qquad 2x_1 - x_2 + 2x_3 \ge 2 \\ &\qquad -x_1 + 2x_2 + 2x_3 \ge 5 \\ &\qquad x_1, x_2, x_3 \ge 0 \end{aligned}$$

by using the dual simplex method. [**Note** that credit will **not** be given for a solution by any other method.]

[6 marks]

Describe how the 'Big M' method could have been used to transform the above linear program into canonical form, and write down a corresponding initial basis. [Do **not** then proceed to solve this linear program.

[4 marks]

4. (a) Let $\{x_0^*, x_1^*, ..., x_n^*\}$ and $\{y_0^*, y_1^*, ..., y_m^*\}$ denote optimal solutions of the following pair of primal and dual linear programs P and D:

P: maximise $x_0 = p_1x_1 + p_2x_2 + ... + p_nx_n$

subject to

$$
\begin{array}{llllll}
a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n & + x_{n+1} & & & & = b_1 \\
a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n & & + x_{n+2} & & & = b_2 \\
... & ... & ... & ... & ... & ... \\
a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n & & & & + x_{n+m} & = b_m \\
x_1, x_2, ..., x_n, x_{n+1}, ..., x_{n+m} \geq 0 & & & & &
\end{array}
$$

D: minimise $y_0 = b_1y_1 + b_2y_2 + ... + b_my_m$

subject to

$$
\begin{array}{llllll}
a_{11}y_1 + a_{21}y_2 + ... + a_{m1}y_m & + y_{m+1} & & & & = p_1 \\
a_{12}y_1 + a_{22}y_2 + ... + a_{m2}y_m & & + y_{m+2} & & & = p_2 \\
... & ... & ... & ... & ... & ... \\
a_{1n}y_1 + a_{2n}y_2 + ... + a_{mn}y_m & & & & + y_{m+n} & = p_n \\
y_1, y_2, ..., y_m, y_{m+1}, ..., y_{m+n} \geq 0 & & & & &
\end{array}
$$

Assuming that x_0^* and y_0^* both exist and are finite, state the *Duality Theorem* including *Complementary Slackness.*

Write down the dual linear program of

P: maximise $x_0 = 3x_1 + x_2 + 2x_3$

subject to

$$
\begin{aligned}
x_1 - x_2 + 2x_3 &\leq 2 \\
2x_1 + x_2 + 2x_3 &\leq 7 \\
-x_1 + x_2 + 2x_3 &\leq 1 \\
x_1,\ x_2,\ x_3 &\geq 0
\end{aligned}
$$

Given that the optimal solution is $x_1^* = 3$, $x_2^* = 1$, $x_3^* = 0$, use complementary slackness to determine which of the constraints of D are satisfied with equality at optimality. **[8 marks]**

(b) The vertices O, A, B, C, D, E of the feasible region for a bicriterion problem with two objectives Z_1 and Z_2 (which are to be maximised) are shown in the following table together with the corresponding values of the objectives.

Vertex:	O	A	B	C	D	E
(x, y):	(0, 0)	(0, 3)	(1, 4)	(4, 2)	(4, 1)	(3, 0)
(Z_1, Z_2):	(0, 0)	(12, -3)	(17, 0)	(12, 14)	(8, 15)	(3, 12)

Which of the the points O, A, B, C, D and E corresponds to an inferior solution and in each such case give a solution to which it is inferior. What is the *Non Inferior Set* (or NIS) for this problem?

Let $Z(w) = (1 - w)Z_1 + wZ_2$. Determine, as a function of w, the set of points of the feasible region of problem P for which $Z(w)$ is maximised. **[8 marks]**

Give a short description of the method of *Goal Programming* as applied to a bicriterion linear program. **[4 marks]**

5. (a) Sketch the state transition diagram for an M/M/1 queueing system in which arrivals occur at a mean rate λ and the server operates at mean rate μ.

Prove that the steady state probability of there being n users in the system ($n = 0, 1, 2, \ldots$) is given by

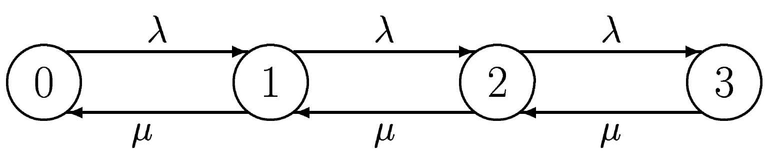
$$p_n = \rho^n(1-\rho) \quad \text{where } \rho = \lambda/\mu < 1.$$

Also find an expressions for L_s the expected number of the users in the system, W_s the expected time spent by a user in the system, W_q the expected waiting time in the queue, and L_q the expected number of the users in the queue.

[**Note**. You are given that $\sum_{n=0}^{\infty} x^n = (1-x)^{-1}$ and $\sum_{n=0}^{\infty} nx^n = x(1-x)^{-2}$ if $|x| < 1$]

[10 marks]

(b) A manufacturing company has a small machine repair shop which can accommodate at most 3 machines in the shop at any one time. Assuming that an adequate model for the repair shop is an M/M/1 queueing system with storage limited to 3 machines the corresponding state transition diagram is



Find, in terms of $\rho = \lambda/\mu$, expressions for p_n, the probability of there being n users in the system, $n = 0, 1, 2, 3$, and also L_s, the expected number of users in the system.

Machines for repair which cannot enter the repair shop (because it already has 3 machines for repair) are repaired by an outside company at a cost of £200 per machine. If $\lambda = 2$ machines for repair per day, $\mu = 2$ repairs per day (i.e. $\rho = 1$) calculate the mean daily cost of outside repair work.

The company is considering upgrading the repair shop by employing extra repair staff at a cost of £85 per day. This will lead to the service rate being doubled (i.e. ρ is halved) but the limitation of at most 3 machines simultaneously in the shop still holds. Obtain, under these new conditions, the daily cost of outside repair work and determine whether the company should go ahead with the upgrade. **[10 marks]**

6. (a) For a single-item continuous review inventory system the Total Cost per Unit time is given, in the usual notation, by $TCU(y) = KD/y + (1/2)hy$, where y is the order size and K, D and h are constants. You are given that the corresponding Economic Order Quantity is $y^* = \sqrt{(2KD/h)}$. Prove that $TCU(y^*) = \sqrt{2KDh}$.

If $K = £100$ per order, $D = 230$ items per week, and $h = £1.1$ per item per week, calculate

the Economic Order Quantity $y*$,

the associated daily cost $TCU(y*)$,

the average stock held, and

the interval T between orders.

Since the working week is 5 days long what order quantity y would you recommend *in practice*?

Derive the formula

$$\frac{TCU(y)}{TCU(y^*)} = \frac{1}{2}\left(\frac{y}{y^*} + \frac{y^*}{y}\right)$$

and use it to determine the range of y for which $TCU(y)$ exceeds $TCU(y^*)$ by no more than 2%. **[15 marks]**

(b) For the problem of part (a) it is noticed that there is some variability in demand and it is found that a reasonable model is that demand follows a Normal distribution with mean 46 units per working week and standard deviation 5 units per working day. If the delivery lead time is 2 working days, determine the mean demand and standard deviation of the demand during the lead time. Use the quasi-static continuous review model to find the level of buffer stock which should be kept to ensure that the probability of a stock-out does not exceed 5%.

[**Note** that you may assume that if Z has a Normal distribution with mean 0 and variance 1 then the probability that Z exceeds 1.64 is approximately 0.05.]

[5 marks]

7 (a) Solve the following Transportation Problem starting from the initial basic feasible solution given.

	P	Q	R
A	[7]	[5] 4	[8] 8
B	[6] 3	[4] 2	[8]
C	[7] 7	[5]	[2]

State, giving reasons for your answer, whether there is an optimal solution corresponding to a different basis? If so give one.

[8 marks]

(b) In a rural district there are four villages V_1, V_2, V_3, V_4 and two schools S_1, S_2. Distances (in miles) between schools and villages are as follows

	V_1	V_2	V_3	V_4
S_1:	5	5	6	2
S_2:	6	3	4	6

The numbers of new school children in each village each year are

V_1	V_2	V_3	V_4
14	16	14	14

The two schools can each accommodate 29 new pupils each year.

Write down, in tableau form, a Transportation Problem to minimise the total distance children must travel to school. Using the North West Corner Rule to find an initial basic feasible solution, solve this Transportation Problem.

It is noticed that children from V_2 are split between the schools. Accordingly, school S_2 is given permission to accept up to 30 new pupils each year. Explain how this new situation may be modelled as a balanced Transportation Problem. Go on to solve this new problem.

[12 marks]