1. (a) Define

 $\begin{array}{rl} x_1 = {\rm Tons} \mbox{ of A produced per week} \\ x_2 = {\rm Tons} \mbox{ of B produced per week} \\ {\rm Problem is to maximise } x_0 = 400 x_1 + 300 x_2 \ (\pounds \mbox{ per week}) \\ {\rm subject to} & 0.3 x_1 + 0.4 x_2 & \leq & 300 \ ({\rm tons \ per \ week}) \\ & x_1 & \geq & 500 \ ({\rm tons \ per \ week}) \\ & x_2 & \geq & 200 \ ({\rm tons \ per \ week}) \\ & x_1, x_2 & \geq & 0 \end{array}$ 

Introduce slack variable  $s_1$ , surplus variables  $s_2, s_3$ , artificial variables  $A_1, A_2$  with associated penalty M, then problem becomes

maximise 
$$x_0 = 400x_1 + 300x_2 - MA_1 - MA_2$$
  
subject to  
 $0.3x_1 + 0.4x_2 + s_1 = 300$   
 $x_1 - s_2 + A_1 = 500$   
 $x_2 - s_3 + A_2 = 200$   
 $x_1, x_2, s_1, s_2, s_3, A_1, A_2 \ge 0$ 

Eliminating  $A_1, A_2$  from the objective function,

$$x_0 = 400x_1 + 300x_2 - M(500 - x_1 + s_2 + 200 - x_2 + s_3)$$
  
=  $(400 + M)x_1 + (300 + M)x_2 - Ms_2 - Ms_3 - 700M$ 

So initial tableau is

	$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	$A_1$	$A_2$	
$x_0$	-(400+M)	-(300+M)	0	M	M	0	0	-700M
$s_1$	0.3	0.4	1	0	0	0	0	300
$A_1$	1	0	0	-1	0	1	0	500
$A_2$	0	1	0	0	-1	0	1	200

(b) Define  $x_1$  = Number of drivers working morning shift

 $x_2 =$  Number of drivers working split shift

 $x_3 =$  Number of drivers working afternoon shift

Problem is to minimise  $x_0 = x_1 + x_2 + x_3$  (drivers)

subject to  $\begin{array}{rcl}
x_1 + x_2 &\geq & 50 \text{ (drivers)} \\
x_1 &\geq & 30 \text{ (drivers)} \\
x_3 &\geq & 30 \text{ (drivers)} \\
x_2 + x_3 &\geq & 45 \text{ (drivers)} \\
x_3 &\geq & 20 \text{ (drivers)} \\
x_1, x_2, x_3 &\geq & 0
\end{array}$ 

Redundant constraints:  $x_3 \ge 20, x_1 \ge 0, x_3 \ge 0.$ 

Evaluating z(x, y) at vertices of feasible region, z(2, 0) = 2, z(4, 0) = 4, z(0, 3) = 12, z(0, 2) = 8, so maximum is z = 12 at (x, y) = (0, 3). At optimality, constraints  $x \ge 0$  and  $3x + 4y \le 12$  are binding;  $y \ge 0$ ,  $x + y \ge 2$  and  $3x + 2y \le 12$  are non-binding.

Constraint  $3x + 2y \le 12$  is redundant.

(b) First choose column with most negative objective row entry as the pivot column. Then, for each positive entry in the pivot column, find the ratio between the entry in the same row in the constants column and the entry in the pivot column. The row for which this ratio is smallest is the pivot row.

The purpose of the ratio test is to ensure that when the pivot operation is carried out, no negative entries are introduced into the constants column, since this would mean that one of the variables took a negative value, and so would violate the nonnegativity constraints.

						$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$						
					$x_0$	-3	-1	-1	0	0	0	0					
					$s_1$	1	1	1	1	0	0	4					
					$s_2$	2	1	0	0	1	0	2					
					$s_3$	-1	1	1	0	0	1	2					
							Í			I							l
	$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$					$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	
$x_0$	0	0.5	-1	0	1.5	0	3		a	<sup>c</sup> 0	0	0.5	-1	0	1.5	0	3
$s_1$	0	0.5	1	1	-0.5	0	3	or	8	<sup>3</sup> 1	0	0.5	1	1	-0.5	0	3
$x_1$	1	0.5	0	0	0.5	0	1		a	$c_1$	1	0.5	0	0	0.5	0	1
$s_3$	0	1.5	1	0	0.5	1	3		8	33	0	1.5	1	0	0.5	1	3

(c) Introducing slack variables  $s_1, s_2, s_3$ , then tableaux are

	$x_1$	$x_2$	$x_3$	51	02	03			$x_1$	$x_2$	$x_3$	51	02	$\overline{o}_3$	
$x_0$	0	1	0	1	1	0	6	$x_0$	0	2	0	0	2	1	6
$x_3$	0	0.5	1	1	-0.5	0	3	_					-1		
$x_1$	1	0.5	0	0	0.5	0	1	$x_1$	1	0.5	0	0	0.5	0	1
$s_3$	0	1	0	-1	1	1	0	$x_3$	0	1.5	1	0	0.5	1	3

So optimal solution is  $x_0 = 6$ , when  $x_1 = 1$ ,  $x_2 = 0$ ,  $x_3 = 3$ . Check constraints:  $x_1 + x_2 + x_3 = 1 + 0 + 3 = 4 \le 4$   $2x_1 + x_2 = 2 + 0 = 2 \le 2$   $-x_1 + x_2 + x_3 = -1 + 0 + 3 = 2 \le 2$  $x_1, x_2, x_3 \ge 0$ 

Basic variables are  $x_1, x_3, s_3$ ; alternative optimal basis  $x_1, x_3, s_1$ . Or vice-versa. Or could give either  $x_1, x_3, s_2$  or  $x_1, x_2, x_3$  as alternative basis.

Evaluating z at vertices gives z(0,0) = 0, z(2,0) = 4, z(0.5,3) = 7, z(0,2) = 6 so maximal value is z = 7 at (x, y) = (0.5, 3).

- (i) Lines -2x + y = 2 and 2x = 4 intersect at (2,6), constraint  $2x + y \le c$  become redundant when line 2x + y = c passes through the same point, so when  $c = 2 \times 2 + 6 = 10$ .
- (ii) Optimum remains at (0.5, 3) until objective line 2x + by = const is parallel to the line 2x + y = 4, that is when b = 1, so within the range  $b \ge 1$  the optimum point remains the same.
- (iii) Optimal solution is affected when k increases so that the line 2x + ky = 4 crosses into the feasible region, which happens when it coincides with the line 2x + y = 4, so when k = 1. That is, k can increase by 1 before solution is affected.
- (b) Dual simplex method appropriate when there are ≥ constraints, and usually for minimisation problems. *Primal feasible* means there are no negative values in the constants column (last column) of the tableau; *dual feasible* means there are no negative values in the objective row (top row) of the tableau. Primal algorithm maintains primal feasibility while working towards dual feasibility also; dual algorithm maintains dual feasibility while working towards primal feasibility also. Once primal and dual feasibility are simultaneously satisfied, the optimum has been reached.
- (c) Introducing surplus variables  $s_1, s_2$  and slack variable  $s_3$ , then tableaux are

			$x_3$				
$x_0$	3	1	1	0	0	0	0
$s_1$	1	-2	-1	1	0	0	-2
$s_2$	-2	1	-1 -1 2	0	1	0	-5
$s_3$	1	1	2	0	0	1	12

	$w_1$	$x_{2}$	23	91	02	03	
$x_0$	1	2	0			0	-5
$s_1$	3	-3	0	1	-1	0	3
$x_3$	2	-1	1	0	-1	0	5
$s_3$	-3	3	1 0	0	2	1	2

So optimal solution is  $x_0 = -5$ , when  $x_1 = 0$ ,  $x_2 = 0$ ,  $x_3 = 5$ .

D: minimise  $y_0 = 7y_1 + y_2 + 2y_3$ 

subject to

Given  $x_1^* = 2, x_2^* = 1$  then

 $2x_1^* + 3x_2^* = 7$ , first constraint satisfied with equality;  $-x_1^* + x_2^* = -1$ , second constraint not satisfied with equality;  $x_1^* = 2$ , third constraint satisfied with equality. Given  $y_1^* = \frac{1}{3}$ ,  $y_2^* = 0$ ,  $y_3^* = \frac{1}{3}$  then  $2y_1^* - y_2^* + y_3^* = 1$ , first constraint satisfied with equality;

 $3y_1^* + y_2^* = 1$ , second constraint satisfied with equality.

Complementary slackness requires that for those constraints not satisfied with equality, the dual variable is zero. For the primal, the second constraint is the only one not satisfied with equality, so since  $y_2^* = 0$ , complementary slackness is satisfied. For the dual, both constraints are satisfied with equality, so complementary slackness is satisfied.

(b) O is inferior to A, B, C, D; D is inferior to B, C.

The NIS consists of the edges AB and BC.

When w = 0, optimal point is A; when w = 1, optimal point is C.

Optimum moves from A to B when

$$19(1 - w) + 2w = 14(1 - w) + 16w$$
  

$$19 - 17w = 14 + 2w$$
  

$$-19w = -5$$
  

$$w = 5/19$$

Optimum moves from B to C when

$$14(1 - w) + 16w = 10(1 - w) + 17w$$
  

$$14 + 2w = 10 + 7w$$
  

$$-5w = -4$$
  

$$w = 4/5$$

 $0 \le w < 5/19$ : point A optimal w = 5/19: edge AB optimal 5/19 < w < 4/5: point B optimal w = 4/5: edge BC optimal  $4/5 < w \le 1$ : point C optimal

The goal program is to minimise  $\Delta = 2d_1^- + 3d_2^-$  subject to  $Z_1 - 2 = d_1^+ - d_1^-$ ,  $Z_2 - 5 = d_2^+ - d_2^-$  and  $d_1^+, d_1^-, d_2^+, d_2^- \ge 0$ , together with all constraints which defined the original feasible region *OABCD*.

- A function f(x) is convex if for all x, y in the domain of f we have  $f(\lambda x + (1 \lambda)y) \leq \lambda f(x) + (1 \lambda)f(y)$  for every  $\lambda \in [0, 1]$ . A mathematical programming problem is convex if it involves the minimisation of a convex function over a convex feasible region.
  - (b) Lagrangean is  $L(x, y; \lambda) = x^2 + y^2 + 3xy + 5x + 10y + \lambda (4x + y 5)$ , so for optimum require

$$L_x = 2x + 3y + 5 + 4\lambda = 0$$
  

$$L_y = 2y + 3x + 10 + \lambda = 0$$
  

$$L_\lambda = 4x + y - 5 = 0$$

To solve for  $x, y, \lambda$ ,

$$3L_x - 2L_y = 5y - 5 + 10\lambda = 0 \Rightarrow y = 1 - 2\lambda$$
  
then  $L_x = 0 \Rightarrow x = -(3y + 5 + 4\lambda)/2 = -(3 - 6\lambda + 5 + 4\lambda)/2 = -(8 - 2\lambda)/2 = -4 + \lambda$   
next,  $L_\lambda = 0 \Rightarrow 4(-4 + \lambda) + (1 - 2\lambda) - 5 = 0 \Rightarrow 2\lambda - 20 = 0 \Rightarrow \lambda = 10$   
hence minimum occurs at  $x^* = 6, y^* = -19$ .

(c)

$$f(x,y) = x^{2} + y^{2} - xy - x + 4$$
  

$$\nabla f = (2x - y - 1, 2y - x)$$

Starting from  $(x_0, y_0) = (0, 0)$ , have  $(\nabla f)_0 = (-1, 0)$ , so search along the line

$$(x,y) = (0,0) + \theta(-1,0) = (-\theta,0)$$
  
so  $f(x,y) = \theta^2 + \theta + 4$   
 $df/d\theta = 2\theta + 1$ 

stationary point at  $\theta = -0.5$ , so  $(x_1, y_1) = (0.5, 0)$ . Hence  $(\nabla f)_1 = (0, -0.5)$ , so search along the line

$$(x,y) = (0.5,0) + \theta(0,-0.5) = (0.5,-0.5\theta)$$
  
so  $f(x,y) = 0.25 + 0.25\theta^2 + 0.25\theta - 0.5 + 4$   
 $= 0.25\theta^2 + 0.25\theta + 3.75$   
 $df/d\theta = 0.5\theta + 0.25$ 

stationary point at  $\theta = -0.5$ , so  $(x_2, y_2) = (0.5, 0.25)$ .

With shortages:

(b) To minimise costs, differentiate TCU with respect to y.

$$\frac{d}{dy}TCU = \frac{-KD}{y^2} + \frac{h}{2}, \text{ so at S.P.s, } \frac{KD}{y^2} = \frac{h}{2} \Rightarrow y^2 = \frac{2KD}{h} \Rightarrow y^* = \sqrt{2KD/h}$$

as required.

$$TCU(y^*) = \frac{KD}{y^*} + \frac{hy^*}{2} = \frac{KD}{\sqrt{2KD/h}} + \frac{h\sqrt{2KD/h}}{2} = \sqrt{\frac{KDh}{2}} + \sqrt{\frac{KDh}{2}} = \sqrt{2KDh}$$

as required.

With K = 80, D = 360, h = 0.64, then  $y^* = \sqrt{2 \times 80 \times 360/0.64} = \sqrt{90000} = 300$  items  $TCU(y^*) = \sqrt{2 \times 80 \times 360 \times 0.64} = \sqrt{36864} = \pounds 192$ Average stock held  $= y^*/2 = 300/2 = 150$  items  $T = y^*/D = 300/360 = 0.83$  weeks.

To find required range of y, set  $\rho = y/y^*$ , then

$$1.03 = (\rho + (1/\rho))/2$$
  

$$2.06\rho = \rho^2 + 1$$
  

$$\rho^2 - 2.06\rho + 1 = 0$$
  

$$\rho = (2.06 \pm \sqrt{2.06^2 - 4})/2 = 1.03 \pm 0.2468$$
  

$$= 0,7832, 1.2768$$

So required range is  $[0.7832y^*, 1.2768y^*] = [235, 383]$  items.

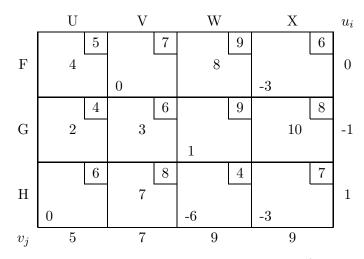
(c) Dimerchiciating 100 with respect to y and w;

$$\frac{d}{dy}TCU = -\frac{KD}{y^2} + \frac{h}{2} - \frac{(h+p)w^2}{2y^2}$$
$$\frac{d}{dw}TCU = -h + \frac{(h+p)w}{y}$$

At SPs, have dTCU/dw = 0, so that w/y = h/(h+p). Also have dTCU/dy = 0, and substituting for w/y gives

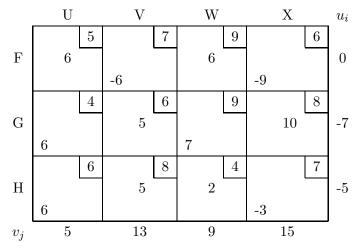
$$\frac{KD}{y^2} = \frac{h}{2} - \frac{(h+p)h^2}{2(h+p)^2}$$
$$= \frac{h}{2} - \frac{h^2}{2(h+p)}$$
$$= \frac{hp}{2(h+p)}$$
$$y^2 = \frac{2(h+p)KD}{hp}$$
$$y^* = \sqrt{2KD(p+h)/ph}$$

.. (a)

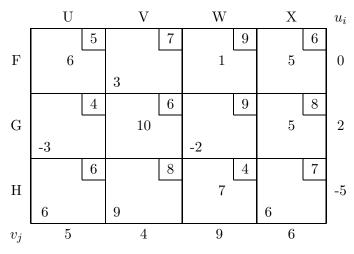


(Compute  $u_i$ ,  $v_j$  using  $u_i + v_j = c_{ij}$  for cells ij in the basis (and  $u_F = 0$ ), then for non-basic cells compute  $\delta_{ij} = c_{ij} - u_i - v_j$ .)

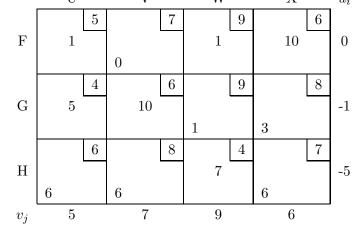
Not optimal, as  $\delta_{FX}$  and  $\delta_{HW}$  are negative. Increase flow through cell with most negative  $\delta$  value, is cell HW, by as much as possible.



Need to increase flow through cell FX by as much as possible.



Need to increase flow through cell GU by as much as possible.

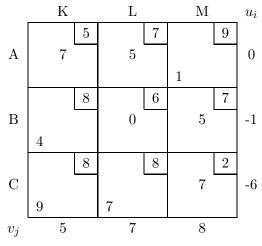


No negative  $\delta$  values, so optimum has been attained.

Have  $\delta_{FV} = 0$ , so there is an alternative optimal basis. Bringing FV into the basis gives the alternative optimal solution

	U	V	W	Х
F	5	1	9	10 6
G	6	9 9	9	8 10
Н	6	8	7	7

(b) Initial basic feasible solution is



No negative  $\delta$  values, so solution is optimal.

If supply at A is increased to 13, then total supply is 25, total demand is 24. To model as a balanced problem, introduce a dummy destination with demand 1, with 'transportation costs' to the dummy destination representing costs of over-production, eg storage costs at the three sources.

34, total supply is 25. To model as a balanced problem, need to introduce a dummy source with supply = 9, with 'transportation costs' from the dummy source being used to represent the cost of failing to meet demand.