MATH461 January 2005 Exam Solutions

1. (a) Define

$x_1 =$ Tons of A produced per week

$x_2 =$ Tons of B produced per week

Problem is to maximise $x_0 = 400x_1 + 300x_2$ (£ per week)

subject to
$$\begin{aligned} 0.3x_1 + 0.4x_2 &\leq 300 \text{ (tons per week)} \\ x_1 &\geq 500 \text{ (tons per week)} \\ x_2 &\geq 200 \text{ (tons per week)} \\ x_1, x_2 &\geq 0 \end{aligned}$$

Introduce slack variable s_1, surplus variables s_2, s_3, artificial variables A_1, A_2 with associated penalty M, then problem becomes

maximise $x_0 = 400x_1 + 300x_2 - MA_1 - MA_2$

subject to
$$\begin{aligned} 0.3x_1 + 0.4x_2 + s_1 &= 300 \\ x_1 - s_2 + A_1 &= 500 \\ x_2 - s_3 + A_2 &= 200 \\ x_1, x_2, s_1, s_2, s_3, A_1, A_2 &\geq 0 \end{aligned}$$

Eliminating A_1, A_2 from the objective function,

$$\begin{aligned} x_0 &= 400x_1 + 300x_2 - M(500 - x_1 + s_2 + 200 - x_2 + s_3) \\ &= (400 + M)x_1 + (300 + M)x_2 - Ms_2 - Ms_3 - 700M \end{aligned}$$

So initial tableau is

	x_1	x_2	s_1	s_2	s_3	A_1	A_2	
x_0	$-(400+M)$	$-(300+M)$	0	M	M	0	0	$-700M$
s_1	0.3	0.4	1	0	0	0	0	300
A_1	1	0	0	-1	0	1	0	500
A_2	0	1	0	0	-1	0	1	200

(b) Define $x_1 =$ Number of drivers working morning shift

$x_2 =$ Number of drivers working split shift

$x_3 =$ Number of drivers working afternoon shift

Problem is to minimise $x_0 = x_1 + x_2 + x_3$ (drivers)

subject to
$$\begin{aligned} x_1 + x_2 &\geq 50 \text{ (drivers)} \\ x_1 &\geq 30 \text{ (drivers)} \\ x_3 &\geq 30 \text{ (drivers)} \\ x_2 + x_3 &\geq 45 \text{ (drivers)} \\ x_3 &\geq 20 \text{ (drivers)} \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Redundant constraints: $x_3 \geq 20$, $x_1 \geq 0$, $x_3 \geq 0$.

2. (a) Feasible region.

Evaluating $z(x, y)$ at vertices of feasible region, $z(2, 0) = 2$, $z(4, 0) = 4$, $z(0, 3) = 12$, $z(0, 2) = 8$, so maximum is $z = 12$ at $(x, y) = (0, 3)$.

At optimality, constraints $x \geq 0$ and $3x + 4y \leq 12$ are binding; $y \geq 0$, $x + y \geq 2$ and $3x + 2y \leq 12$ are non-binding.

Constraint $3x + 2y \leq 12$ is redundant.

(b) First choose column with most negative objective row entry as the pivot column. Then, for each positive entry in the pivot column, find the ratio between the entry in the same row in the constants column and the entry in the pivot column. The row for which this ratio is smallest is the pivot row.

The purpose of the ratio test is to ensure that when the pivot operation is carried out, no negative entries are introduced into the constants column, since this would mean that one of the variables took a negative value, and so would violate the non-negativity constraints.

(c) Introducing slack variables s_1, s_2, s_3, then tableaux are

	x_1	x_2	x_3	s_1	s_2	s_3	
x_0	-3	-1	-1	0	0	0	0
s_1	1	1	1	1	0	0	4
s_2	$\boxed{2}$	1	0	0	1	0	2
s_3	-1	1	1	0	0	1	2

	x_1	x_2	x_3	s_1	s_2	s_3	
x_0	0	0.5	-1	0	1.5	0	3
s_1	0	0.5	$\boxed{1}$	1	-0.5	0	3
x_1	1	0.5	0	0	0.5	0	1
s_3	0	1.5	1	0	0.5	1	3

or

	x_1	x_2	x_3	s_1	s_2	s_3	
x_0	0	0.5	-1	0	1.5	0	3
s_1	0	0.5	1	1	-0.5	0	3
x_1	1	0.5	0	0	0.5	0	1
s_3	0	1.5	$\boxed{1}$	0	0.5	1	3

	x_1	x_2	x_3	s_1	s_2	s_3	
x_0	0	1	0	1	1	0	6
x_3	0	0.5	1	1	-0.5	0	3
x_1	1	0.5	0	0	0.5	0	1
s_3	0	1	0	-1	1	1	0

	x_1	x_2	x_3	s_1	s_2	s_3	
x_0	0	2	0	0	2	1	6
s_1	0	-1	0	1	-1	-1	0
x_1	1	0.5	0	0	0.5	0	1
x_3	0	1.5	1	0	0.5	1	3

So optimal solution is $x_0 = 6$, when $x_1 = 1$, $x_2 = 0$, $x_3 = 3$.

Check constraints:
$$\begin{aligned} x_1 + x_2 + x_3 &= 1+0+3 = 4 \leq 4 \\ 2x_1 + x_2 &= 2+0 = 2 \leq 2 \\ -x_1 + x_2 + x_3 &= -1+0+3 = 2 \leq 2 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Basic variables are x_1, x_3, s_3; alternative optimal basis x_1, x_3, s_1. Or vice-versa. Or could give either x_1, x_3, s_2 or x_1, x_2, x_3 as alternative basis.

3. (a) Feasible region:

Evaluating z at vertices gives $z(0,0) = 0$, $z(2,0) = 4$, $z(0.5,3) = 7$, $z(0,2) = 6$ so maximal value is $z = 7$ at $(x, y) = (0.5, 3)$.

(i) Lines $-2x + y = 2$ and $2x = 4$ intersect at $(2, 6)$, constraint $2x + y \leq c$ become redundant when line $2x + y = c$ passes through the same point, so when $c = 2 \times 2 + 6 = 10$.

(ii) Optimum remains at $(0.5, 3)$ until objective line $2x + by = const$ is parallel to the line $2x + y = 4$, that is when $b = 1$, so within the range $b \geq 1$ the optimum point remains the same.

(iii) Optimal solution is affected when k increases so that the line $2x + ky = 4$ crosses into the feasible region, which happens when it coincides with the line $2x + y = 4$, so when $k = 1$. That is, k can increase by 1 before solution is affected.

(b) Dual simplex method appropriate when there are $\geq$ constraints, and usually for minimisation problems. *Primal feasible* means there are no negative values in the constants column (last column) of the tableau; *dual feasible* means there are no negative values in the objective row (top row) of the tableau. Primal algorithm maintains primal feasibility while working towards dual feasibility also; dual algorithm maintains dual feasibility while working towards primal feasibility also. Once primal and dual feasibility are simultaneously satisfied, the optimum has been reached.

(c) Introducing surplus variables s_1, s_2 and slack variable s_3, then tableaux are

	x_1	x_2	x_3	s_1	s_2	s_3	
x_0	3	1	1	0	0	0	0
s_1	1	-2	-1	1	0	0	-2
s_2	-2	1	$\boxed{-1}$	0	1	0	-5
s_3	1	1	2	0	0	1	12

	x_1	x_2	x_3	s_1	s_2	s_3	
x_0	1	2	0	0	1	0	−5
s_1	3	−3	0	1	−1	0	3
x_3	2	−1	1	0	−1	0	5
s_3	−3	3	0	0	2	1	2

So optimal solution is $x_0 = -5$, when $x_1 = 0$, $x_2 = 0$, $x_3 = 5$.

1. (a) Dual is

D: minimise $y_0 = 7y_1 + y_2 + 2y_3$

subject to
$$\begin{aligned} 2y_1 - y_2 + y_3 &\geq 1 \\ 3y_1 + y_2 &\geq 1 \\ y_1, y_2, y_3 &\geq 0 \end{aligned}$$

Given $x_1^* = 2$, $x_2^* = 1$ then
$2x_1^* + 3x_2^* = 7$, first constraint satisfied with equality;
$-x_1^* + x_2^* = -1$, second constraint not satisfied with equality;
$x_1^* = 2$, third constraint satisfied with equality.

Given $y_1^* = \frac{1}{3}$, $y_2^* = 0$, $y_3^* = \frac{1}{3}$ then
$2y_1^* - y_2^* + y_3^* = 1$, first constraint satisfied with equality;
$3y_1^* + y_2^* = 1$, second constraint satisfied with equality.

Complementary slackness requires that for those constraints not satisfied with equality, the dual variable is zero. For the primal, the second constraint is the only one not satisfied with equality, so since $y_2^* = 0$, complementary slackness is satisfied. For the dual, both constraints are satisfied with equality, so complementary slackness is satisfied.

(b) O is inferior to A, B, C, D; D is inferior to B, C.

The NIS consists of the edges AB and BC.

When $w = 0$, optimal point is A; when $w = 1$, optimal point is C.

Optimum moves from A to B when
$$\begin{aligned} 19(1-w) + 2w &= 14(1-w) + 16w \\ 19 - 17w &= 14 + 2w \\ -19w &= -5 \\ w &= 5/19 \end{aligned}$$

Optimum moves from B to C when
$$\begin{aligned} 14(1-w) + 16w &= 10(1-w) + 17w \\ 14 + 2w &= 10 + 7w \\ -5w &= -4 \\ w &= 4/5 \end{aligned}$$

$0 \leq w < 5/19$: point A optimal
$w = 5/19$: edge AB optimal
$5/19 < w < 4/5$: point B optimal
$w = 4/5$: edge BC optimal
$4/5 < w \leq 1$: point C optimal

The goal program is to minimise $\Delta = 2d_1^- + 3d_2^-$ subject to $Z_1 - 2 = d_1^+ - d_1^-$, $Z_2 - 5 = d_2^+ - d_2^-$ and $d_1^+, d_1^-, d_2^+, d_2^- \geq 0$, together with all constraints which defined the original feasible region $OABCD$.

3. (a) A set S is convex if for every $x, y \in S$ and $\lambda \in [0,1]$ we have $\lambda x + (1-\lambda)y \in S$.

A function $f(x)$ is convex if for all x, y in the domain of f we have $f(\lambda x+(1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$ for every $\lambda \in [0,1]$.

A mathematical programming problem is convex if it involves the minimisation of a convex function over a convex feasible region.

(b) Lagrangean is $L(x, y; \lambda) = x^2 + y^2 + 3xy + 5x + 10y + \lambda\,(4x + y - 5)$, so for optimum require

$$\begin{aligned} L_x &= 2x + 3y + 5 + 4\lambda &= 0 \\ L_y &= 2y + 3x + 10 + \lambda &= 0 \\ L_\lambda &= 4x + y - 5 &= 0 \end{aligned}$$

To solve for x, y, λ,

$$3L_x - 2L_y = 5y - 5 + 10\lambda = 0 \Rightarrow y = 1 - 2\lambda$$

$$\text{then } L_x = 0 \Rightarrow x = -(3y + 5 + 4\lambda)/2 = -(3 - 6\lambda + 5 + 4\lambda)/2 = -(8 - 2\lambda)/2 = -4 + \lambda$$

$$\text{next, } L_\lambda = 0 \Rightarrow 4(-4 + \lambda) + (1 - 2\lambda) - 5 = 0 \Rightarrow 2\lambda - 20 = 0 \Rightarrow \lambda = 10$$

$$\text{hence minimum occurs at } x^* = 6, y^* = -19.$$

(c)

$$\begin{aligned} f(x, y) &= x^2 + y^2 - xy - x + 4 \\ \nabla f &= (2x - y - 1,\ 2y - x) \end{aligned}$$

Starting from $(x_0, y_0) = (0, 0)$, have $(\nabla f)_0 = (-1, 0)$, so search along the line

$$\begin{aligned} (x, y) &= (0, 0) + \theta(-1, 0) &= (-\theta, 0) \\ \text{so } f(x, y) &= \theta^2 + \theta + 4 \\ df/d\theta &= 2\theta + 1 \end{aligned}$$

stationary point at $\theta = -0.5$, so $(x_1, y_1) = (0.5, 0)$.

Hence $(\nabla f)_1 = (0, -0.5)$, so search along the line

$$\begin{aligned} (x, y) &= (0.5, 0) + \theta(0, -0.5) &= (0.5, -0.5\theta) \\ \text{so } f(x, y) &= 0.25 + 0.25\theta^2 + 0.25\theta - 0.5 + 4 \\ &= 0.25\theta^2 + 0.25\theta + 3.75 \\ df/d\theta &= 0.5\theta + 0.25 \end{aligned}$$

stationary point at $\theta = -0.5$, so $(x_2, y_2) = (0.5, 0.25)$.

6. (a) Stock level against time.

With shortages:

(b) To minimise costs, differentiate TCU with respect to y.

$$\frac{d}{dy}TCU = \frac{-KD}{y^2} + \frac{h}{2}, \text{ so at S.P.s, } \frac{KD}{y^2} = \frac{h}{2} \Rightarrow y^2 = \frac{2KD}{h} \Rightarrow y^* = \sqrt{2KD/h}$$

as required.

$$TCU\left(y^*\right) = \frac{KD}{y^*} + \frac{hy^*}{2} = \frac{KD}{\sqrt{2KD/h}} + \frac{h\sqrt{2KD/h}}{2} = \sqrt{\frac{KDh}{2}} + \sqrt{\frac{KDh}{2}} = \sqrt{2KDh}$$

as required.

With $K = 80$, $D = 360$, $h = 0.64$, then
$y^* = \sqrt{2 \times 80 \times 360/0.64} = \sqrt{90000} = 300$ items
$TCU\left(y^*\right) = \sqrt{2 \times 80 \times 360 \times 0.64} = \sqrt{36864} = £192$
Average stock held $= y^*/2 = 300/2 = 150$ items
$T = y^*/D = 300/360 = 0.83$ weeks.

To find required range of y, set $\rho = y/y^*$, then

$$\begin{aligned}
1.03 &= (\rho + (1/\rho))/2 \\
2.06\rho &= \rho^2 + 1 \\
\rho^2 - 2.06\rho + 1 &= 0 \\
\rho &= \left(2.06 \pm \sqrt{2.06^2 - 4}\right)/2 = 1.03 \pm 0.2468 \\
&= 0, 7832, 1.2768
\end{aligned}$$

So required range is $[0.7832y^*, 1.2768y^*] = [235, 383]$ items.

(c) Differentiating TCU with respect to y and w,

$$\begin{aligned}
\frac{d}{dy}TCU &= -\frac{KD}{y^2} + \frac{h}{2} - \frac{(h+p)w^2}{2y^2} \\
\frac{d}{dw}TCU &= -h + \frac{(h+p)w}{y}
\end{aligned}$$

At SPs, have $dTCU/dw = 0$, so that $w/y = h/(h+p)$. Also have $dTCU/dy = 0$, and substituting for w/y gives

$$\begin{aligned}
\frac{KD}{y^2} &= \frac{h}{2} - \frac{(h+p)h^2}{2(h+p)^2} \\
&= \frac{h}{2} - \frac{h^2}{2(h+p)} \\
&= \frac{hp}{2(h+p)} \\
y^2 &= \frac{2(h+p)KD}{hp} \\
y^* &= \sqrt{2KD(p+h)/ph}
\end{aligned}$$

7. (a)

	U	V	W	X	u_i
F	$\boxed{5}$ 4	$\boxed{7}$ 0	$\boxed{9}$ 8	$\boxed{6}$ -3	0
G	$\boxed{4}$ 2	$\boxed{6}$ 3	$\boxed{9}$ 1	$\boxed{8}$ 10	-1
H	$\boxed{6}$ 0	$\boxed{8}$ 7	$\boxed{4}$ -6	$\boxed{7}$ -3	1
v_j	5	7	9	9	

(Compute u_i, v_j using $u_i + v_j = c_{ij}$ for cells ij in the basis (and $u_F = 0$), then for non-basic cells compute $\delta_{ij} = c_{ij} - u_i - v_j$.)

Not optimal, as δ_{FX} and δ_{HW} are negative. Increase flow through cell with most negative δ value, ie cell HW, by as much as possible.

	U	V	W	X	u_i
F	$\boxed{5}$ 6	$\boxed{7}$ -6	$\boxed{9}$ 6	$\boxed{6}$ -9	0
G	$\boxed{4}$ 6	$\boxed{6}$ 5	$\boxed{9}$ 7	$\boxed{8}$ 10	-7
H	$\boxed{6}$ 6	$\boxed{8}$ 5	$\boxed{4}$ 2	$\boxed{7}$ -3	-5
v_j	5	13	9	15	

Need to increase flow through cell FX by as much as possible.

	U	V	W	X	u_i
F	$\boxed{5}$ 6	$\boxed{7}$ 3	$\boxed{9}$ 1	$\boxed{6}$ 5	0
G	$\boxed{4}$ -3	$\boxed{6}$ 10	$\boxed{9}$ -2	$\boxed{8}$ 5	2
H	$\boxed{6}$ 6	$\boxed{8}$ 9	$\boxed{4}$ 7	$\boxed{7}$ 6	-5
v_j	5	4	9	6	

Need to increase flow through cell GU by as much as possible.

	U	V	W	X	u_i
F	$\boxed{5}$ 1	$\boxed{7}$ 0	$\boxed{9}$ 1	$\boxed{6}$ 10	0
G	$\boxed{4}$ 5	$\boxed{6}$ 10	$\boxed{9}$ 1	$\boxed{8}$ 3	-1
H	$\boxed{6}$ 6	$\boxed{8}$ 6	$\boxed{4}$ 7	$\boxed{7}$ 6	-5
v_j	5	7	9	6	

No negative δ values, so optimum has been attained.

Have $\delta_{FV} = 0$, so there is an alternative optimal basis. Bringing FV into the basis gives the alternative optimal solution

	U	V	W	X
F	$\boxed{5}$	$\boxed{7}$ 1	$\boxed{9}$ 1	$\boxed{6}$ 10
G	$\boxed{4}$ 6	$\boxed{6}$ 9	$\boxed{9}$	$\boxed{8}$ 10
H	$\boxed{6}$	$\boxed{8}$	$\boxed{4}$ 7	$\boxed{7}$

(b) Initial basic feasible solution is

	K	L	M	u_i
A	$\boxed{5}$ 7	$\boxed{7}$ 5	$\boxed{9}$ 1	0
B	$\boxed{8}$ 4	$\boxed{6}$ 0	$\boxed{7}$ 5	-1
C	$\boxed{8}$ 9	$\boxed{8}$ 7	$\boxed{2}$ 7	-6
v_j	5	7	8	

No negative δ values, so solution is optimal.

If supply at A is increased to 13, then total supply is 25, total demand is 24. To model as a balanced problem, introduce a dummy destination with demand 1, with 'transportation costs' to the dummy destination representing costs of over-production, eg storage costs at the three sources.

If demand at M then increases by 10, no need for dummy destination, total demand is 34, total supply is 25. To model as a balanced problem, need to introduce a dummy source with supply = 9, with 'transportation costs' from the dummy source being used to represent the cost of failing to meet demand.