THE UNIVERSITY
of LIVERPOOL

1. (a) A chemical company produces two products, A and B. The company can sell A at a profit of £400 per ton, and B at a profit of £300 per ton. The production processes also yield toxic wastes, with each ton of A produced resulting in 0.3 tons of toxic waste and each ton of B produced resulting in 0.4 tons of toxic waste. Government regulations require that the company produce not more than 300 tons of toxic waste per week. The company is contracted to provide a particular customer with 500 tons of A and 200 tons of B per week, and any production in excess of this can be sold to other customers at the same price. Formulate (but do **not** solve) a linear program to determine the amounts of A and B which the company should produce per week in order to maximise its profit.

   One method of solving the above problem is to introduce artificial variables and apply the 'big M' method. Proceed with the 'big M' method as far as writing down the initial tableau. (Note that you are **not** required to perform any simplex iterations.) [10 marks]

   (b) A bus company operates buses on several routes between 7:30am and 7:30pm. The timetables are such that the minimum numbers of drivers required at particular times of day are as follows.

Time of day	Minimum number of drivers required
7:30am—9:30am	50
9:30am–4:00pm	30
4:00pm–6:00pm	45
6:00pm–7:30pm	20

   On any given day, each driver will work one of the following three shifts:

   Morning shift: 7:30am–1:30pm

   Split shift: 7:30am–9:30am and 4:00pm–6:00pm

   Afternoon shift: 1:30pm–7:30pm.

   Formulate (but do **not** solve) a linear program to determine the numbers of drivers working morning, split and afternoon shifts so as to minimise the total number of drivers required subject to all routes being covered.

   Identify any redundant constraints in your linear program. [10 marks]

2. (a) Sketch the feasible region for the linear program

maximise $z = x + 4y$
subject to

$$\begin{aligned} x + y &\geq 2 \\ 3x + 4y &\leq 12 \\ 3x + 2y &\leq 12 \\ x, y &\geq 0 \end{aligned}$$

Determine the optimal solution and its value.
State which constraints are binding at optimality and which are non-binding.
Which (if any) constraints are redundant? [8 marks]

(b) In the primal simplex method, explain how the choice of pivot element is made. What is the purpose of the *ratio test* in this context? [4 marks]

(c) Use the simplex method to solve

maximise $x_0 = 3x_1 + x_2 + x_3$
subject to

$$\begin{aligned} x_1 + x_2 + x_3 &\leq 4 \\ 2x_1 + x_2 &\leq 2 \\ -x_1 + x_2 + x_3 &\leq 2 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Check that your solution satisfies the constraints. What are the basic variables in your optimal solution? State whether there is an alternative optimal basis, and if so then write one down. [8 marks]

3. (a) Sketch the feasible region of the problem

maximise $z = 2x + by$
subject to

$$\begin{aligned} 2x + ky &\leq 4 \\ -2x + y &\leq 2 \\ 2x + y &\leq c \\ x, y &\geq 0 \end{aligned}$$

when c and k take the values 4 and 0 respectively. Identify the optimal solution for $b = 2$ and give the opimal solution value. Answer the following questions in each of which just one of c, k and b is varied from the values given above.

(i) To what value must c increase before $2x + y \leq c$ becomes redundant?

(ii) Find the range of b for which the optimal solution remains optimal.

(iii) By how much can k increase without the optimal solution being affected?

[10 marks]

(b) Describe when the dual simplex method is appropriate for solving linear programs. Define the terms *primal feasible* and *dual feasible* in terms of a simplex tableau, and explain how primal and dual feasibility are affected by the operations of the primal and dual simplex algorithms. [4 marks]

(c) Solve the linear program

maximise $x_0 = -3x_1 - x_2 - x_3$
subject to

$$\begin{aligned} -x_1 + 2x_2 + x_3 &\geq 2 \\ 2x_1 - x_2 + x_3 &\geq 5 \\ x_1 + x_2 + 2x_3 &\leq 12 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

by using the dual simplex method. (Note that credit will **not** be given for a solution by any other method.) [6 marks]

4. (a) Write down the dual linear program D of the problem
P: maximise $x_0 = x_1 + x_2$
subject to

$$\begin{aligned} 2x_1 + 3x_2 &\leq 7 \\ -x_1 + x_2 &\leq 1 \\ x_1 &\leq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Given the optimal solution of P is $x_1^* = 2$, $x_2^* = 1$ and the optimal solution of D is $y_1^* = \frac{1}{3}$, $y_2^* = 0$, $y_3^* = \frac{1}{3}$ establish which primal and which dual constraints are satisfied as equalities. Hence verify that the complementary slackness relations are satisfied. [8 marks]

(b) The vertices O, A, B, C, D of the feasible region for a bicriterion problem with two objectives Z_1 and Z_2 (which are to be maximised) are shown in the following table together with the corresponding values of the objectives.

Vertex:	O	A	B	C	D
(x, y):	(0, 0)	(1, 4)	(4, 2)	(4, 1)	(3, 0)
(Z_1, Z_2):	(2, 2)	(19, 2)	(14, 16)	(10, 17)	(5, 14)

Which of the points O, A, B, C and D corresponds to an inferior solution? In each such case give a solution to which it is inferior. What is the *Non Inferior Set* (or NIS) for this problem?

Let $Z(w) = (1 - w)Z_1 + wZ_2$. Determine, as a function of w for $0 \leq w \leq 1$, the set of points of the feasible region for which $Z(w)$ is maximised.

Suppose the goals for Z_1 and Z_2 are $G_1 = 2$ and $G_2 = 5$ respectively. Explain how to formulate a goal program in which the penalties for undershooting goals G_1 and G_2 are 2 and 3 per unit, respectively, and where there is no penalty for overshooting.

[12 marks]

5. (a) Define the terms *convex set* and *convex function*. In what circumstances is a mathematical programming problem said to be convex? [4 marks]

(b) Consider the problem

minimise $f(x, y) = x^2 + y^2 + 3xy + 5x + 10y$

subject to $4x + y = 5$

Given that there exists a unique local minimum point (x^*, y^*) which is also the global minimum point, use the Lagrangean method to solve the above problem. (Note that credit will **not** be given for a solution by any other method.)

[8 marks]

(c) For the problem

minimise $f(x, y) = x^2 + y^2 - xy - x + 4$

carry out two iterations of the steepest descent algorithm, starting from the initial point $(x_0, y_0) = (0, 0)$. [8 marks]

6. (a) Sketch the graph of 'stock level' against 'time' for a single-item static (i.e. constant demand) continuous review inventory model with instantaneous replacement, showing at least two cycles. Next, sketch a modified version of your graph for the case in which shortages are allowed. [4 marks]

(b) For a single-item continuous static review inventory system the Total Cost per Unit time is given, in the usual notation, by $TCU(y) = KD/y + (1/2)hy$, where y is the order size and K, D and h are positive constants. Prove that the corresponding Economic Order Quantity is $y^* = \sqrt{(2KD/h)}$ and that $TCU(y^*) = \sqrt{2KDh}$.

If $K = £80$ per order, $D = 360$ items per week, and $h = 64$ pence per item per week, calculate

the Economic Order Quantity $y*$,
the associated weekly cost $TCU(y*)$,
the average stock held, and
the interval T between orders.

Given the formula

$$\frac{TCU(y)}{TCU(y^*)} = \frac{1}{2}\left(\frac{y}{y^*} + \frac{y^*}{y}\right)$$

and the data above, determine the range of y for which $TCU(y)$ exceeds $TCU(y^*)$ by no more than 3%. [12 marks]

(c) For a single-item continuous static review inventory system in which shortages are allowed, the Total Cost per Unit time is given, in the usual notation, by

$$TCU(y, w) = \frac{KD}{y} + \frac{hy}{2} - hw + \frac{(h+p)w^2}{2y}$$

where y is the order size, w is the size of shortage permitted, and K, D, h and p are positive constants. Prove that the corresponding Economic Order Quantity is $y^* = \sqrt{2KD(p+h)/ph}$. [4 marks]

7. (a) Solve the following Transportation Problem starting from the initial basic feasible solution given.

	U	V	W	X
F	4 $\boxed{5}$	$\boxed{7}$	8 $\boxed{9}$	$\boxed{6}$
G	2 $\boxed{4}$	3 $\boxed{6}$	$\boxed{9}$	10 $\boxed{8}$
H	$\boxed{6}$	7 $\boxed{8}$	$\boxed{4}$	$\boxed{7}$

State, giving reasons for your answer, whether there is an optimal solution corresponding to a different basis. If so give one. [12 marks]

(b) Use the North West Corner Rule to provide an initial basic feasible solution for the following Transportation Problem. Is this basis optimal?

	K	L	M	Supply
A	$\boxed{5}$	$\boxed{7}$	$\boxed{9}$	12
B	$\boxed{8}$	$\boxed{6}$	$\boxed{7}$	5
C	$\boxed{8}$	$\boxed{8}$	$\boxed{2}$	7
Demand	7	5	12	

Suppose now that the supply at A is increased to 13. Explain how the resulting unbalanced problem may be modelled as a balanced Transportation Problem. [Do **not** go on to solve the problem.]

Suppose further that the demand of M increases by 10. Explain how the resulting unbalanced problem may be modelled as a balanced Transportation Problem. [Do **not** go on to solve the problem.] [8 marks]

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