1. (a) A small paint factory produces both exterior and interior house paints. Two raw materials, A and B, are used to manufacture the paints. Each ton of exterior paint is made up of 0.3 tons of A and 0.7 tons of B, while each ton of interior paint is made up of 0.6 tons of A and 0.4 tons of B. The maximum availabilty of A is 16 tons per day; the maximum availability of B is 12 tons per day.

A market survey has established that daily demand for interior paint cannot exceed that for exterior paint by more than 2 tons. The survey also shows that the maximum demand for interior paint is limited to 5 tons daily. The company can sell exterior paint at a profit of £3000 per ton and interior paint at a profit of £2000 per ton, and wishes to know how much of each paint to produce in order to maximise total daily profit.

Formulate this problem as a linear program. (Do **not** go on to solve it.)

[8 marks]

(b) Sketch the feasible region of the following problem, and use your graph (do **not** use tableaux) to find the optimal solution.

maximise $x_0 = x_1 + 2x_2$

subject to

$$\begin{aligned} x_1 + x_2 &\leq 5 \\ -x_1 + x_2 &\leq 1 \\ x_1 - 2x_2 &\leq 2 \\ x_1 + 2x_2 &\leq 9 \\ x_1 &\geq 1 \\ x_2 &\leq 3 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Which, if any, of the constraints are redundant?

If all constraints remain as above, but the objective function is now taken to be $x_0 = ax_1 + 2x_2$ for some constant a, then for what range of a values will the co-ordinates (x_1, x_2) at which the maximal x_0 value is attained remain as before?

[12 marks]

2. (a) In the primal simplex method, explain how the choice of pivot element is made. What is the purpose of the *ratio test* in this context?

[4 marks]

(b) Use the primal simplex method to solve the linear program

maximise $x_0 = 2x_1 + 3x_2 + 5x_3$
subject to

$$\begin{aligned} x_1 + x_2 - x_3 &\leq 5 \\ 6x_1 - 2x_2 - 9x_3 &\leq 4 \\ x_1 + x_2 + 4x_3 &\leq 10 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

[8 marks]

(c) Solve the following linear program by introducing artificial variables and applying the 'big M' method. (Solution by any other method will **not** receive credit.)

maximise $x_0 = -3x_1 - x_2 + 3x_3$
subject to

$$\begin{aligned} x_1 + x_2 &\geq 4 \\ 2x_1 + 5x_3 &= 10 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

[8 marks]

3. (a) Write down the dual linear program D of

P: maximise $x_0 = 2x_1 + 2x_2 + 3x_3$

subject to

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &\leq 6 \\ 2x_1 + x_2 + 3x_3 &\leq 4 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Solve the dual problem D graphically, and hence write down the maximal value of x_0 for the primal problem P.

Write down the complementary slackness conditions for this problem, and use them to find the values of x_1, x_2, x_3 at which the maximal value of x_0 is achieved for the primal problem P.

[12 marks]

(b) Use the dual simplex algorithm to solve the following linear program. (Solution by any other method will **not** receive credit.)

maximise $x_0 = -4x_1 - 3x_2 - 3x_3$

subject to

$$\begin{aligned} x_1 + x_2 + x_3 &\geq 10 \\ x_1 + 2x_2 + 2x_3 &\geq 4 \\ x_1 - x_2 - 2x_3 &\geq 4 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

[8 marks]

4. (a) For a bicriterion maximisation problem with objectives Z_1 and Z_2, explain what the *weighting method* is, and why it is used.

[4 marks]

(b) Consider the bicriterion problem
maximise $\{Z_1, Z_2\}$
subject to

$$\begin{aligned} x &\geq 1 \\ x + y &\leq 8 \\ 2x + y &\leq 11 \\ 3x - 2y &\leq 6 \\ x, y &\geq 0 \end{aligned}$$

(i) Sketch the feasible region for this problem.

[3 marks]

(ii) If $Z_1 = x + 2y$ and $Z_2 = 3x + y$, state the values of x, y, Z_1, Z_2 at each vertex of the feasible region. Identify which of these vertices are inferior solutions and hence find the Non-Inferior Set (NIS).

[3 marks]

(iii) Let $Z(w) = (1 - w)Z_1 + wZ_2$, where $0 \leq w \leq 1$. Determine, as a function of w, the set of points of the feasible region for which $Z(w)$ is maximised.

[4 marks]

(iv) Sketch the feasible region for this problem in **objective space**.

[2 marks]

(v) Find the Non-Inferior Set for the problem with the extra condition that x and y must be integers.

[4 marks]

5. (a) Consider a single-item static continuous review model in which the set-up cost is K per order, the demand rate is D units per unit time and the holding cost is h per unit per unit time. If the order size is y units, then the Total Cost per Unit time is given by

$$TCU(y) = \frac{KD}{y} + \frac{hy}{2}$$

and the Economic Order Quantity y^* is given by $y^* = \sqrt{2KD/h}$.

Calculate the values of y^* and $TCU(y^*)$ if $K = £80$ per order, $D = 100$ units per day and $h = 3$ pence per unit per day.

If it is learned that orders must be in multiples of 50 units, determine the order size which would lead to minimal cost.

If the company's working week is 7 days long, then considering the time between orders, what order size would you finally suggest, and why?

[6 marks]

(b) Consider now a single-item static continuous review model in which shortages are permitted. Suppose the set-up cost is K per order, the demand rate is D units per unit time, the holding cost is h per unit per unit time, and the shortage cost is p per unit per unit time. Sketch a graph of stock level against time, and hence show that the Total Cost per Unit time is given by

$$TCU(y, w) = \frac{KD}{y} + \frac{h(y-w)^2}{2y} + \frac{pw^2}{2y},$$

where the order size is y units and shortages of up to w units are permitted to occur.

By differentiating $TCU(y, w)$ with respect to w and setting the derivative equal to zero, show that the values of y and w which minimise cost satisfy

$$\frac{w}{y} = \frac{h}{h+p}.$$

Hence by differentiating $TCU(y, w)$ with respect to y, find expressions in terms of h, p, K and D for the values of y and w which minimise cost.

Investigate and comment on the behaviour of the system (i) as p increases towards infinity; and (ii) as p decreases towards zero.

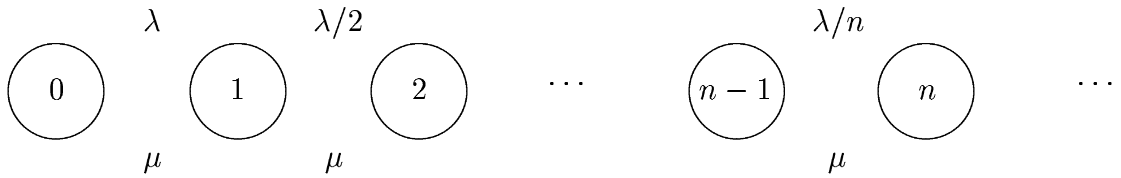
[14 marks]

6. (a) Sketch the state transition diagram for an $(M/M/1)$ queueing system in which arrivals occur at mean rate λ and the server operates at mean rate μ.

Explain how the $(M/M/2)$ queueing system differs from the $(M/M/1)$ queueing system.

[4 marks]

(b) A queueing system is characterised by the following state transition diagram.



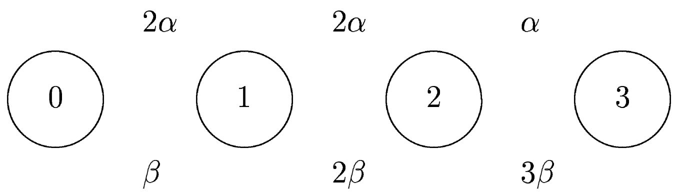
Thus if i represents the number of users in the system $(i = 0, 1, 2, \ldots)$, the mean arrival rate when in state i is $\lambda/(i+1)$ and the mean service rate is μ. Defining $\rho = \lambda/\mu$, show that for $n = 0, 1, 2, \ldots$, the steady state probability of being in state n is given by

$$p_n = \frac{\rho^n \exp(-\rho)}{n!}$$

$\left[\text{You may assume the formula } \sum_{n=0}^{\infty} \frac{x^n}{n!} = \exp(x).\right]$

[8 marks]

(c) Two repairmen are attending 3 machines in a workshop. The time taken to repair a machine is exponentially distributed with mean $1/\alpha$. After being repaired, the time for which a machine will operate before breaking down again is exponentially distributed with mean $1/\beta$. Thus if the 'state of the system' is the number of machines in operation, then the state transition diagram is as shown below.



[**Question 6 continued overleaf.**]

If $\alpha = 3$ per hour and $\beta = 4$ per hour, then calculate the following quantities, assuming that the system is in a steady state.

(i) The probabilities p_0, p_1, p_2, p_3, where p_n is the probability that there are n machines in operation;

(ii) The average number of machines in operation L_S;

(iii) The average length of time for which all 3 machines are out of operation during an 8 hour day.

[8 marks]

7. (a) A company has four factories producing a certain product which is to be shipped to three distribution centres. In units of thousands of items, factories A, B, C, D produce 15, 22, 18, 9 per week, respectively. Distribution centres 1 and 2 require 20 thousand items per week each, while distribution centre 3 requires 24 thousand items per week. The distances (in kilometres) from each factory to each distribution centre are as follows.

		Distribution centre		
		1	2	3
	A	100	150	40
Factory	B	60	100	130
	C	80	40	80
	D	100	120	120

The freight cost for each shipment is £500 plus £0.50 per kilometre. It is required to know how much should be shipped from each factory to each distribution centre in order to minimise the total shipping cost. Formulate this problem as a transportation problem and use the North West Corner Rule to find an initial basic feasible solution. Is your initial solution optimal?

Suppose now that the demand at distribution centre 1 rises to 24 thousand items per week. Explain how this unbalanced problem could be modelled as a balanced transportation problem.

[8 marks]

(b) Solve the following transportation problem starting from the initial basic feasible solution given.

[**Question 7 continued overleaf.**]

	A	B	C	D
U	[6] 10	[4] 5	[5]	[15]
V	[12]	[20] 4	[8] 11	[4]
W	[10]	[4]	[5] 12	[10] 3

What is the cost of (i) the given initial solution; (ii) your optimal solution? Does an alternative optimal solution exist? If so, give one.

[12 marks]

/End