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1. Say how you would distinguish between the three types of conic (elliptic, hyperbolic and parabolic).

Determine the type of the conic with equation

$$
7 x^{2}-8 x y+y^{2}-20 x+14 y-23=0
$$

Find its centre (if it is central), the equations of its axes and (if appropriate) the equations of its asymptotes.

Determine the canonical form of its equation and find its eccentricity.
Sketch the conic on graph paper after drawing precisely its axes and marking precisely the foci and the points of intersection of the conic with its axes. [Use the portrait position with a scale of 1 unit $=1 \mathrm{~cm}$. Choose the origin near the centre of the graph paper and the coordinate axes parallel to the edges of the paper.]
[25 marks]
2. Define the term cusp for a parametric curve, and describe how you would determine the order of a cusp.

A parametric curve is defined by

$$
\mathbf{r}(t)=\left(t^{2},-t^{4}+t^{5}\right) \quad(t \in \mathbf{R})
$$

Show that the curve has exactly one non-regular point and that this is a cusp of order 2. Determine the cuspidal tangent line.

Determine the point $P$ on the curve, not at the cusp, at which the tangent is parallel to the $x$-axis.

Show that the curve has exactly one linear point $Q$ and that this is a simple inflexion.

Plot and draw the curve on graph paper for $-0.8 \leq t \leq 1.3$ using a scale of 1 unit $=10 \mathrm{~cm}$. Plot about 13 points including those given by $t=-0.8,0,0.8,1,1.3$. Mark on your sketch the points $P$ and $Q$ and the cuspidal tangent line. [Use the landscape position with the $x$-axis in the centre and the $y$-axis 1 cm . from the left hand side of the paper.]
[25 marks]

# THE UNIVERSITY of LIVERPOOL 

3. Show that the cycloid

$$
\mathbf{r}(t)=(t-\sin t, 1-\cos t) \quad(t \in \mathbf{R})
$$

has speed

$$
\left|\mathbf{r}^{\prime}(t)\right|=2\left|\sin \frac{t}{2}\right|
$$

Deduce, or prove otherwise, that $\mathbf{r}$ has a non-regular point wherever $t=2 \pi n$ with $n$ an integer. Show that each such point is a simple cusp with a vertical cuspidal tangent line.

Show also that the tangent to the curve is parallel to the $x$-axis whenever $t=(2 n+1) \pi$ with $n$ an integer.

Obtain formulae for the arc-length of the curve between the point given by $t=0$ and that given by $t=t_{0}$ in the two cases (i) $0 \leq t_{0} \leq 2 \pi$, (ii) $2 \pi \leq t_{0} \leq 4 \pi$.

Plot and draw the curve for $0 \leq t \leq 4 \pi$, paying special attention to points near $t=2 \pi$. Plot about 13 points including those given by $t=0, \pi, 2 \pi, 3 \pi, 4 \pi$. [Use the scale 1 unit $=2 \mathrm{~cm}$. with the graph paper in the landscape position, the $x$-axis in the centre and the $y$-axis at the extreme left hand side of the paper.]
[25 marks]

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4. Define the term vertex for a parametric curve r. State the relationship between vertices of $\mathbf{r}$ and non-regular points of the evolute of $\mathbf{r}$.

Show that the parabola

$$
\mathbf{r}(t)=\left(t^{2}, 2 t\right) \quad(t \in \mathbf{R})
$$

is regular.
Show that the curvature $\kappa$ of this curve is given by

$$
\kappa(t)=-\frac{1}{2\left(1+t^{2}\right)^{3 / 2}}
$$

Find a formula for $\kappa^{\prime}(t)$ and hence determine any points of the parabola which are vertices.

Show that the evolute of $\mathbf{r}$ has parametric equation

$$
\mathbf{r}_{*}(t)=\left(2+3 t^{2},-2 t^{3}\right) .
$$

Determine the non-regular points of $\mathbf{r}_{*}$ without further calculation.
Plot and draw the original curve and the evolute on the same diagram for $-1.8 \leq t \leq 1.8$, paying special attention to points near $t=0$. Mark and label on your sketch the points on both curves corresponding to the values of the parameter $t=0, \pm 0.5, \pm 1, \pm 1.5, \pm 1.8$. [Use the scale 1 unit $=1 \mathrm{~cm}$. with the graph paper in the portrait position, the $y$-axis on the extreme left hand side and the $x$-axis in the centre of the paper.]
[25 marks]

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5. Define the term undulation for a parametric curve.

Show that the limaçon

$$
z(t)=(2+\cos t) e^{i t} \quad(0 \leq t \leq 2 \pi)
$$

is regular.
Using the formula

$$
\kappa=-\frac{\operatorname{im}\left(z^{\prime} \overline{z^{\prime \prime}}\right)}{\left|z^{\prime}\right|^{3}}
$$

or otherwise, show that the curvature $\kappa$ of this limaçon is given by

$$
\kappa(t)=\frac{6+6 \cos t}{(5+4 \cos t)^{3 / 2}} .
$$

Obtain also a formula for $\kappa^{\prime}(t)$.
Show that the limaçon has an undulation. Show further that the curve has precisely four vertices.

Sketch the curve on polar graph paper (supplied) using a scale of 1 unit $=2$ cm . and plotting points at intervals of $20^{\circ}$. Mark carefully on your sketch the undulation and the vertices.
6. Define the term singular-set envelope.

Verify that, for any $\lambda \in \mathbf{R}$, the line given parametrically by

$$
\mathbf{r}_{\lambda}(t)=(t \cos \lambda+(1-t) \cos 3 \lambda, t \sin \lambda-(1-t) \sin 3 \lambda) \quad(t \in \mathbf{R})
$$

passes through the points $(\cos \lambda, \sin \lambda)$ and $(\cos 3 \lambda,-\sin 3 \lambda)$ of the unit circle.
Determine the singular set of the family $\left\{\mathbf{r}_{\lambda}\right\}$ and show that the family has an envelope of the form

$$
\mathbf{e}(u)=\left(\frac{3}{2} \cos u-\frac{1}{2} \cos 3 u, \frac{3}{2} \sin u+\frac{1}{2} \sin 3 u\right) .
$$

Show that $\mathbf{e}$ has four non-regular points at $u=\frac{\pi}{4}, \frac{3 \pi}{4}, \frac{5 \pi}{4}, \frac{7 \pi}{4}$. [You may find the formulae $\cos 3 u=4 \cos ^{3} u-3 \cos u$, $\sin 3 u=3 \sin u-4 \sin ^{3} u$ useful.]

Using polar graph paper (supplied) and a scale of 1 unit $=2 \mathrm{~cm}$., draw accurately at least 18 of the lines $\mathbf{r}_{\lambda}$ for values of $\lambda$ in the range $0 \leq \lambda \leq 2 \pi$. Mark the non-regular points of the envelope on your sketch.

# THE UNIVERSITY of LIVERPOOL 

7. Let $\Pi$ be a projective curve and $\Gamma$ its affine part. State a sufficient condition for $\Gamma$ to be bounded and say whether or not your condition is also necessary.

Consider the projective curve $\Pi$ given by

$$
x^{4}+x^{2} y z-y^{3} z+y^{4}=0
$$

in homogeneous coordinates $(x: y: z)$. Show that $\Pi$ has a unique singular point at $(0: 0: 1)$.

Let $\Gamma$ be the affine part of $\Pi$, with $z=0$ as the line at infinity. Show that $\Gamma$ is bounded.

Show further that the origin $(x, y)=(0,0)$ is a node of $\Gamma$ of multiplicity 3 and find the tangent lines at the node.

Using the lines $y=t x$ through the origin, prove that the curve has a parametrisation

$$
\mathbf{r}(t)=(x, y)=\left(\frac{t^{3}-t}{1+t^{4}}, \frac{t^{4}-t^{2}}{1+t^{4}}\right)
$$

Show that the point $(0,1)$ lies on $\Gamma$, but that there is no value of $t$ for which $\mathbf{r}(t)=(0,1)$.

Using the parametrisation, plot the curve on graph paper after first drawing the nodal tangent lines. [Use the landscape position and a scale of 1 unit $=10$ cm . with the $x$-axis 3 cm . from the bottom of the paper. You should plot points for values of $t$ in the ranges -5 to $-1,-1$ to 0,0 to 1 and 1 to 5 , and consider also what happens as $t \longrightarrow \pm \infty$.]
[25 marks]

