1. Say how you would distinguish between the three types of conic (elliptic, hyperbolic and parabolic). Determine the type of the conic with equation

$$2x^2 - 4xy + 5y^2 + 4x - 16y - 22 = 0.$$

Find its centre (if it is central), the equations of its axes and (if appropriate) the equations of its asymptotes.

Determine the canonical form of its equation and find its eccentricity.

Sketch the conic on graph paper after drawing precisely its axes and marking precisely the foci and the points of intersection of the conic with its axes. [Use the landscape position with a scale of 1 unit = 1 cm. Choose the origin near the centre of the graph paper and the coordinate axes parallel to the edges of the paper.]

[25 marks]

2. Define the term *simple inflexion* for a parametric curve.

A parametric curve is defined by

$$\mathbf{r}(t) = (t^2, t^3 - t^4) \quad (t \in \mathbf{R}).$$

Show that the curve has exactly one non-regular point. Find the order of the cusp. Determine the cuspidal tangent line.

Determine the point P on the curve, not at the cusp, at which the tangent is parallel to the x-axis.

Show that the curve has one linear point Q and that this point is a simple inflexion.

Plot and draw the curve on graph paper for $-1.0 \leq t \leq 1.2$ using a scale of 1 unit = 10 cm. Plot about 13 points including that given by $t = 0$. Indicate the points P and Q and the cuspidal tangent line. [Use the portrait position with the x-axis 4 cm. from the top and the y-axis near the left hand side of the paper.]

[25 marks]

3. Show that the parametric curve

$$\mathbf{r}(t) = (8\cos^3 t, 8\sin^3 t) \quad (0 \le t \le 2\pi)$$

has speed

$$|\mathbf{r}'(t)| = 12|\sin 2t|.$$

Deduce, or prove otherwise, that $\mathbf{r}$ has precisely four non-regular points at $t = 0$, $t = \frac{\pi}{2}$, $t = \pi$ and $t = \frac{3\pi}{2}$. Show that each of these points is a cusp with either a horizontal or a vertical tangent line.

Obtain formulae for the arc-length of the curve between the point given by $t = 0$ and that given by $t = t_0$ in the two cases $0 \le t_0 \le \frac{\pi}{2}$, $\frac{\pi}{2} \le t_0 \le \pi$.

Plot and draw the curve for $0 \le t \le 2\pi$, paying special attention to points near one of the cusps. Plot points at intervals of $\frac{\pi}{8}$. Mark on your sketch the points corresponding to the values of the parameter $t = 0, \frac{\pi}{4}, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ and the cuspidal tangent line. [Use the scale 1 unit = 1 cm. with the graph paper in the portrait position and the origin near the centre of the paper.] [25 marks]

4. Show that the parametric curve

$$\mathbf{r}(t) = (\cosh t, \sinh t) \quad (t \in \mathbf{R})$$

is regular.

Show that the curvature κ of this curve is given by

$$\kappa(t) = -\frac{1}{(\sinh^2 t + \cosh^2 t)^{3/2}}$$

and that the evolute has parametric equation

$$\mathbf{r}_*(t) = (2\cosh^3 t, -2\sinh^3 t).$$

Find the non-regular points of $\mathbf{r}_*$.

Plot and draw the original curve and the evolute on the same diagram for $-1.2 \le t \le 1.2$, paying special attention to points near $t = 0$. Mark on your sketch the points on both curves corresponding to the values of the parameter $t = 0, \pm 0.8, \pm 1.2$. [Use the scale 1 unit = 1 cm. with the graph paper in the portrait position, the x-axis in the centre and the y-axis near the left hand side of the paper.] [25 marks]

5. Show that the limaçon

$$z(t) = (1 + 2\cos t)e^{it} \quad (0 \le t \le 2\pi)$$

is regular, but possesses a multiple point corresponding to the values of the parameter t where $1 + 2\cos t = 0$. Compute these values.

Using the formula

$$\kappa = -\frac{\mathrm{im}(z'\,\overline{z''})}{|z'|^3},$$

or otherwise, show that the curvature κ of the limaçon is given by

$$\kappa(t) = \frac{9 + 6\cos t}{(5 + 4\cos t)^{3/2}}.$$

Obtain also a formula for $\kappa'(t)$.

Show that the limaçon has no inflexions and precisely two vertices.

State the *four vertex theorem* and explain why the limaçon described above does not contradict the theorem.

Plot the curve on polar graph paper (supplied) using a scale of 1 unit = 2 cm. and plotting points at intervals of 20°. Mark carefully on your sketch the multiple point and the vertices. [25 marks]

6. Verify that, for any $\lambda \in \mathbf{R}$, the line given parametrically by

$$\mathbf{r}_\lambda(t) = (2\cos\lambda - 1 - 2t\sin\lambda, 2\sin\lambda + t(2\cos\lambda - 1)) \quad (t \in \mathbf{R})$$

passes through the point $(2\cos\lambda - 1, 2\sin\lambda)$ and is perpendicular to the line joining this point to the origin.

Determine the singular set of the family $\{\mathbf{r}_\lambda\}$ and show that the family has an envelope of the form

$$\mathbf{e}(u) = \frac{1}{2 - \cos u}(5\cos u - 4, 3\sin u).$$

Using a scale of 1 unit = 3.5 cm., draw accurately on graph paper at least 18 of the lines $\mathbf{r}_\lambda$ for values of λ in the range $0 \le \lambda \le 2\pi$. [Use polar graph paper (supplied) in the landscape position and start by drawing a circle with centre at (-1,0) (i.e. 3.5 cm. to the left of the central point) and radius 2 units; the points $(2\cos\lambda - 1, 2\sin\lambda)$ then lie on this circle.] [25 marks]

7. Let Π be a projective curve. Define the *affine part* Γ of Π. State (in terms of the points of Π at infinity) a condition which ensures that Γ has no asymptotes.

Consider the projective curve Π given by

$$z(x^2 - y^2) - y^3 = 0$$

in homogeneous coordinates $(x : y : z)$. Show that Π has a unique singular point at $(0 : 0 : 1)$.

Show further that Π meets the line $z = 0$ in a unique point $P = (1 : 0 : 0)$ and that the tangent line to Π at P is $z = 0$.

The affine part Γ of Π, with $z = 0$ as the line at infinity, is given by the equation

$$x^2 - y^2 - y^3 = 0.$$

Show that Γ has no asymptotes. Show further that the origin $(x, y) = (0, 0)$ is a node of Γ and find the tangent lines at the node.

Using the lines $x = ty$ through the origin, prove that the curve has a parametrisation

$$\mathbf{r}(t) = (x, y) = (t(t^2 - 1), t^2 - 1)).$$

Using the parametrisation, plot the curve on graph paper in the range $-2 \leq t \leq 2$ after first drawing the nodal tangent lines. [Use the landscape position and a scale of 1 unit = 2 cm. with the origin near the centre of the paper.]

[25 marks]