1.

(a) Let f be the bilinear form on $\mathbb{R}^2$ defined by

$$f\big((x_1, y_1), (x_2, y_2)\big) = -2x_1x_2 + x_1y_2 - y_1x_2 + y_1y_2.$$

Let $u_1 = (1, 3), u_2 = (-1, 2)$. Compute $f(u_1, u_1), f(u_1, u_2), f(u_2, u_1), f(u_2, u_2)$. Find the matrix A of f relative to the basis $\{u_1, u_2\}$.

(b) Also find the matrix B of f relative to the basis $\{v_1, v_2\}$, where $v_1 = (-2, -1), v_2 = (3, 4)$.

(c) Find the change of basis matrix P from $\{u_1, u_2\}$ to $\{v_1, v_2\}$ and check that $B = P^T AP$.

(d) Now consider the quadratic form

$$q(x, y, z) = -2x^2 + 4xy + 2y^2 - 2xz + 2z^2.$$

Give the matrix C representing q with respect to the standard basis.

(e) Find a diagonal matrix D equivalent to C and the matrix Q which describes the change of basis from the standard basis to the basis in which q is diagonal.

(f) What are the rank and signature of q? What type of quadric surface is described by the equation $q(x, y, z) = 3$?

[20 marks]

2.

(a) Let V be a vector space. Say what it means for vectors $v_1, \ldots, v_k$ to form a *basis* of V.

(b) Let U be the subspace of $\mathbb{R}^3$ spanned by

$$u_1 = (1, 1, -1), \quad u_2 = (2, 1, 1), \quad u_3 = (1, -1, 5).$$

Find the dimension of U.

(c) Let W be the subspace of $\mathbb{R}^3$ spanned by

$$w_1 = (1, 2, -4), \quad w_2 = (3, 1, 3), \quad w_3 = (2, -1, 7).$$

Show that $U = W$.

(d) Now let $V = \mathrm{Pol}_3(\mathbb{R})$ be the vector space of polynomials in x, of degree at most three, with real coefficients. Let

$$U = \{ax^3 + bx^2 + cx + d : d + a = b + 2c\}.$$

Show that U is a subspace of V.

(e) Similarly,

$$W = \{(2a + 2b)x^3 + 2ax^2 + (3a - b)x - 2b : a, b \in \mathbb{R}\}$$

is a subspace of V [you do not need to show this]. What are the dimensions of each of $U, W, U \cap W$ and $U + W$? Is it true or false that $U + W = U \oplus W$?

[20 marks]

3.

(a) Let V and W be finite-dimensional vector spaces, and let $\varphi : V \to W$ be linear. Define the *rank* and *nullity* of φ. State the rank and nullity theorem.

(b) Let $V = \mathrm{Pol}_3(\mathbb{R})$ (the vector space of polynomials in x, of degree at most three, with real coefficients), and let $W = \mathbb{R}^3$. Define $\varphi : V \to W$ by

$$\varphi(ax^3 + bx^2 + cx + d) = (a + b + d, 2a + b + c - d, 3a + 2b + c).$$

Show that φ is a linear map and compute its rank and nullity. Is φ an isomorphism?

(c) Now let $V = \mathrm{Pol}_2(\mathbb{R})$ be the vector space of polynomials in x, of degree at most *two*, with real coefficients. Let the linear map $\varphi : V \to V$ be defined by

$$\varphi(ax^2 + bx + c) = (4c - a - 2b)x^2 + (2a + 4b - 8c)x + b - 3c.$$

[You need not show that φ is linear.] Find M, the matrix representation of φ with respect to the basis $\{x^2, x, 1\}$.

(d) What are the eigenvalues and eigenvectors of M?

(e) Is M diagonalizable? If yes, find an invertible matrix P and a diagonal matrix D such that $D = P^{-1}MP$.

[20 marks]

4.

(a) Define the terms: *group, homomorphism, injective, surjective.*

(b) Let G be a group. Which of the following statements are true, and which are false? Give counterexamples to any false statements. You do not need to prove the true statements.

(i) Let $a, b \in G$. Then $b^{-1}ab = a$.

(ii) Let $a, b \in G$, and suppose that $ab = e$ (where e is the identity element of G). Then $b = a^{-1}$.

(iii) If $a, b, c \in G$, then $(ab)c = a(bc)$.

(iv) If $a \in G$ with $a^3 = e$, then $a = e$.

(c) Let G be the group of real numbers under addition, and let H be the group of invertible 2×2 matrices with real entries, under matrix multiplication. [You need not show that these are groups]. Let $\varphi : G \to H$ be defined by

$$\varphi(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

Show that φ is a homomorphism. State, giving reasons, whether φ is injective. State, giving reasons, whether φ is surjective.

(d) The following is a partially completed group table. Fill in the missing entries. You must justify why each choice of entry is the only one possible.

*	A	B	C	D	E	F
A	?	C	?	?	?	?
B	?	?	B	?	?	?
C	?	?	?	?	?	?
D	F	?	?	?	?	A
E	?	?	?	A	C	B
F	?	D	?	?	?	?

(e) Name a known group with the same group table.

[20 marks]

5.

(a) Say what it means for a function $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ to be an *isometry* of the plane.

(b) Show that the set of isometries of the plane forms a group under composition. (You may use the fact that the composition of functions is associative.)

(c) Let ϱ be the anticlockwise rotation around 0 by 60 degrees, and let σ be reflection in the y-axis. Write the isometry $\varphi = \varrho \circ \sigma$ as a single rotation, reflection or translation.

Is φ a linear map from $\mathbb{R}^2$ to itself? If yes, also give the matrix representation of φ with respect to the standard basis of $\mathbb{R}^2$.

(d) From lectures, you know that every isometry φ of the plane can be written as a composition $\varphi = \tau \circ \alpha$, where α is either a reflection in a line through the origin, or a rotation around the origin, and τ is a translation.

Let φ be the anticlockwise rotation around the point $(1, 0)$ by ninety degrees. Find τ and α as above such that $\varphi = \tau \circ \alpha$.

[20 marks]

6. Let the linear map $\varphi : \mathbb{R}^4 \to \mathrm{Pol}_3(\mathbb{R})$ be given by

$$\varphi(a, b, c, d) = (a + b - c)x^3 + (a + b + d)x^2 + (b + c + 3d)x + a + 2b + 3d.$$

[You need not show that φ is linear. Recall that $\mathrm{Pol}_3(\mathbb{R})$ is the vector space of polynomials in x, of degree at most three, with real coefficients.]

(a) Find a basis for U, the image of φ, and a basis for W, the kernel of φ. What are the rank and the nullity of φ?

(b) Consider the basis $B = (x^3 + x^2 + 1, x^3 + x^2 + x + 2, x - x^3, 1)$ of $\mathrm{Pol}_3(\mathbb{R})$. [You need not show that this is a basis.] Find the change-of-basis matrix P from the standard basis $(x^3, x^2, x, 1)$ to B. Compute the inverse matrix P^{-1}.

(c) Compute the matrix representation M of the linear map φ with respect to the standard basis of $\mathbb{R}^4$ and the basis B of $\mathrm{Pol}_3(\mathbb{R})$.

(d) Find a basis C of $\mathbb{R}^4$ such that the matrix M' of φ with respect to C and B is

$$M' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

[20 marks]