M744 2005 Solutions.

1. (a) To say that $\{v_1, v_2, \ldots, v_n\}$ *spans* V means that every element in V can be written as a linear combination

$$\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n.$$

[1 mark]

Now, to show that W is a subspace of V, first note that the zero vector (0,0,0) is in W because the sum of its coordinates is 0+0+0=0. If now (x_1, y_1, z_1) and (x_2, y_2, z_2) are in W, then by definition $x_1 + y_1 + z_1 = 0$ and also $x_2 + y_2 + z_2 = 0$. So since

$$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2),$$

we add the three coordinates of this vector to obtain

$$(x_1 + x_2)+, (y_1 + y_2) + (z_1 + z_2) = (x_1 + y_1 + z_1) + (x_2 + y_2 + z_2) = 0 + 0 = 0$$

Finally, if (x, y, z) is in W (so that $x + y + z + 0$) and λ is any real number, then $\lambda(x, y, z) = (\lambda x, \lambda y, \lambda z)$. Since

$$\lambda x + \lambda y, +\lambda z = \lambda(x + y + z) = \lambda \cdot 0 = 0$$

We have therefore shown that W is a subspace of V. [3 marks]

When we take the vectors $(1, 0, -1), (1, 2, 1)$ and $(2, -2, -4)$, it is clear that the first two are independent, so we investigate what happens if we write

$$(2, -2, -4) = \lambda(1, 0, -1) + \mu(1, 2, 1).$$

This leads to three equations: $2 = \lambda + \mu$, $-2 = 2\mu$ and $-4 = -\lambda + \mu$. We see that $\mu = -1$ and so $\lambda = 3$. Since these equations have non-zero solutions, the third vector depends on the first two so U has basis $(1, 0, -1)$ and $(1, 2, 1)$ and dimension 2. [2 marks]

Now

$$\begin{aligned} W &= \{(x, y, z) : x + y + z = 0\} \\ &= \{(x, y, z) : z = -y - x\} \\ &= \{(x, y, -y - x)\} \\ &= \{x(1, 0, -1) + y(0, 1, -1)\} \end{aligned}$$

Since $(1,0,-1)$ and $(0,1,1)$ are clearly linearly independent, they are a basis for W so W also has dimension 2. [2 marks]

Now if (x,y,z) is in $U \cap W$, then $z = -y - x$ and so $(x, y, -y-x)$ is a linear combination of $(1,0,-1)$ and $((1,2,1)$:

$$(x, y, -y-x) = \lambda(1,0,-1) + \mu(1,2,1)$$

this gives $x = \lambda + \mu$, $y = 2\mu$ and $-y - x = -\lambda + \mu$. Thus $\mu = y/2$ and $\lambda = x - y/2$ (from the first two equations). The third then gives

$$-y - x = -\lambda + \mu = -x + y/2 + y/2 = -x + y$$

it follows that $y = 0$, so vectors of the form $(x, 0, -x)$ are in $U \cap V$ showing that this space has dimension 1. Since $U \cap V \neq \{0\}$, it follows that $\mathbf{R}^3$ is not the direct sum of U and V. [2 marks]

(b) Since $L(1) = x = 0\cdot 1 + 1 \cdot x + 0 \cdot x^2 + 0 \cdot x^3$, the entries in the first column of M are 0, 1, 0, 0. Similarly, we have $L(x) = 1$, $L(x^2) = x^3$ and $L(x^3) = x^2$. It follows that the matrix M is

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

[2 marks]

We next compute $\det(\lambda I - M)$ to get

$$\begin{aligned}
\det(\lambda I - M) &= \\
&= \det \begin{pmatrix} \lambda & -1 & 0 & 0 \\ -1 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & -1 \\ 0 & 0 & -1 & \lambda \end{pmatrix} \\
&= \lambda \det \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & -1 \\ 0 & -1 & \lambda \end{pmatrix} - (-1)\det \begin{pmatrix} -1 & 0 & 0 \\ 0 & \lambda & -1 \\ 0 & -1 & \lambda \end{pmatrix} \\
&= \lambda(\lambda(\lambda^2-1)) - (-1)(-(\lambda^2-1)) \\
&= (\lambda^2-1)(\lambda^2-1) \\
&= (\lambda^2-1)^2
\end{aligned}$$

It follows that M has two repeated eigenvalues, namely 1 (twice) and -1 (twice). [4 marks]

When $\lambda = 1$, a vector $v = a+bx+cx^2+dx^3$ is an eigenvector if $L(v) = v$, so $b+ax+dx^2+cx^3 = a+bx+cx^2+dx^3$. This occurs precisely if $b = a$ and $d = c$, so the eigenvectors are the polynomials of the form $a+ax+cx^2+cx^3$. [2 marks]

When $\lambda = -1$, a vector $v = a+bx+cx^2+dx^3$ is an eigenvector if $L(v) = -v$, so $b+ax+dx^2+cx^3 = -a-bx-cx^2-dx^3$. This occurs precisely if $b = -a$ and $d = -c$, so the eigenvectors are the polynomials of the form $a-ax+cx^2-cx^3$. [2 marks]

2. The kernel of f is the set of vectors v such that $f(v) = 0$. The image of f is the range of values taken by f.The rank of f is the dimension of im f and the nullity of f is the dimension of its kernel. [4 marks]

Now to show that ker f is a subspace, we check the standard three requirements:

First 0 is in ker f because $f(0) = 0$;

Next if u, v are in ker f then $f(u) = 0 = f(v)$. Then since f is a linear map, $f(u+v) = f(u)+f(v) = 0+0 = 0$, so u_+v is in ker f;

Finally, if u is in ker f (so $f(v) = 0$) and λ is any real number $f(\lambda v) = \lambda f(v) = \lambda \cdot 0 = 0$, since f is linear.

We have therefore shown that ker f is a subspace of V.

[3 marks]

The matrix of the given linear map is

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 2 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

To find a basis for the image of f, we need to find a basis for the space spanned by the columns of A: the vectors

$$(1,1,2,0); \quad (1,1,0,0); \quad (1,2,1,0); \text{ and } (1,1,2,0)$$

Clearly the last equals the first, so the only question is whether the third is a linear combination of the first 2. Consider

$$(1,2,1,0) = \lambda(1,1,2,0) + \mu(1,1,0,0)$$

This gives $1 = \lambda + \mu$, $1 = 2\lambda$, $2 = \lambda + \mu$ and $0 = 0$. Clearly the first and third are inconsistent, so they have no solution, so the third vector is not a linear combination of the first two. We deduce that $(1, 1, 2, 0)$, $(1, 1, 0, 0)$ and $(1, 2, 1, 0)$ are a basis for the image and so the rank of f is 3. [5 marks]

The kernel is the solution set for the equations $Au = 0$, giving

$$x + y + z + t = 0; \quad x + y + 2z + t = 0; \quad 2x + z + 2t = 0.$$

It is clear that if we subtract the first two equations, we obtain $z = 0$. Rewriting then gives $x+y+t = 0$ (twice) and $2x+2t = 0$. Thus $t = -x$ and $y = 0$ so the solution set consists of vectors of the form $(x, 0, 0, -x) = x(1, 0, 0, -1)$. This is clearly a one dimensional space spanned by the vector $(1, 0, 0, 1)$ so the nullity is 1.

[5 marks]

To decide whether $\mathbf{R}^4$ is a direct sum of the kernel and the image of f or not, we try to find a u with $f(u) = 0$ and $u = f(v)$. Thus u is of the form $x(1, 0, 0, -1)$ and also u is in the image of f so

$$u = \lambda(1, 1, 2, 0) + \mu(1, 1, 0, 0) + \nu(1, 2, 1, 0)$$

Since all vectors in the image of f have zero fourth coordinate,the only vector common to ker f and im f is that with $x = 0$ so the intersection of ker f and im f is $\{0\}$. Thus the sum of ker f and im f has dimension 4 and so must equal $\mathbf{R}^4$. It follows that $\mathbf{R}^4$ is the direct sum of ker f and im f.

[3 marks]

3. The dual space is defined to be the set of all linear maps from V to $\mathbf{R}$. Given θ, ϕ in V^*, we can define $\theta + \phi$ by $(\theta + \phi)(x) = \theta(x) + \phi(x)$. Similarly, for λ in $\mathbf{R}$, we define $(\lambda\theta)(x) = \lambda(\theta(x))$.

Given a basis $\{x_1, x_2, \ldots, x_n\}$ for V, we define ϕ_i as the unique linear map which maps x_i to 1, but all other basis elements to 0. To prove this gives a dual basis, suppose first that f is any linear map from V to $\mathbf{R}$. Let λ_j be that scalar which f maps x_j to (so that $\lambda_j = f(x_j)$). Then for any j the map $\lambda_1\phi_1 + \cdots \lambda_n\phi_n$ takes x_j to λ_j (since $\phi_i(x_j) = 0$ for $i \neq j$). Thus the maps f and $\lambda_1\phi_1 + \cdots \lambda_n\phi_n$ agree in their action on a basis for V so must be equal and the vectors $\phi_1, \ldots \phi_n$ span V^*. Now to check linear independence, suppose that $\lambda_1\phi_1 + \cdots \lambda_n\phi_n = 0$. Then, for any x_j, $(\lambda_1\phi_1 + \cdots \lambda_n\phi_n)(x_j) = 0$ we also know that $(\lambda_1\phi_1 + \cdots \lambda_n\phi_n)(x_j) = \lambda_j$, so each λ_j would then be zero. Thus $\{\phi_1, \ldots, \phi_n\}$ is a basis for V^*. [7 marks]

Thus we have that

$$\begin{array}{lll} \phi_1(v_1) = 1; & \phi_1(v_2) = 0 & \phi_1(v_3) = 0 \\ \phi_1(v_2) = 0; & \phi_2(v_2) = 1 & \phi_2(v_3) = 0 \\ \phi_1(v_3) = 0; & \phi_3(v_2) = 0 & \phi_3(v_3) = 1. \end{array}$$

[1 mark]

Now if $\phi_1(x,y,z) = a_1x + b_1y + c_1z$, we obtain $a_1 + b_1 + c_1 = 1$, $a_1 + 2b_1 + 4c_1 = 0$ and $a_1 - b_1 + c_1 = 0$. We now solve these equations for a_1, b_1, c_1 to get $2a_1 + 2c_1 = 1$ (so $c_1 = 1/2 - a_1$). We can now re-write the first two to say $b_1 + 1/2 = 1$ (so $b_1 = 1/2$) and $a_1 + 4c_1 = -1$ (so $a_1 = +1$ and $c_1 = -1/2$), so that $\phi_1(x,y,x) = x + y/2 - z/2$. Similar calculations are carried out to determine ϕ_2: we solve

$$a_2 + b_2 + c_2 = 0, \quad a_2 + 2b_2 + 4c_2 = 1, \text{ and } a_2 - b_2 + c_2 = 0$$

These give $a_2 = -1/3, b_2 = 0$ and $c_2 = 1/3$ so that $\phi_2(x,y,x) = -x/3 + z/3$. For ϕ_3, we solve

$$a_3 + b_3 + c_3 = 0, \quad a_3 + 2b_3 + 4c_3 = 0, \text{ and } a_3 - b_3 + c_3 = 1.$$

This time the solution is $a_3 = 1/3, b_3 = -1/2$ and $c_3 = 1/6$ so that ϕ_3 is given by $\phi_3(x,y,z) = x/3 - y/2 + z/6$. [5 marks]

Finally

$$\begin{aligned} \phi_1(3,2,1) &= 3 + 2/2 - 1/2 = 7/2; \\ \phi_2(3,2,1) &= -3/3 + 1/3 = -2/3; \\ \phi_3(3,2,1) &= 3/3 - 2/2 + 1/6 = 1/6. \end{aligned}$$

[2 marks]

Finally, to express the map$f(x,y,z) = x + 2y + 3z$ in terms of ϕ_1, ϕ_2 and ϕ_3, we use the proof given in the answer to the first part of the question. Thus, we let λ_1 be the scalar which f maps $v_1 = (1,1,1)$ to, namely 6, similarly we take λ_2 to be $f(1,2,4) = 17$ and λ_3 to be $f(1,-1,1) = 2$. Thus the required combination is

$$\begin{aligned} 6\phi_1 & +17\phi_2 + 2\phi_3 \\ = & \ 6(x + y/2 - z/2) + 17(-x/3 + z/3) + 2(x/3 - y/2 + z/6) \\ = & \ x + 2y + 3z \end{aligned}$$

as required. [5 marks]

4. We are given that $f((x_1,x_2),(y_1,y_2)) = x_1y_1 + 2x_1y_2 + x_2y_2$. Thus

$$\begin{aligned} f((2,2),(2,2)) &= 2\cdot 2 + 2\cdot 2\cdot 2 + 2\cdot 2 = 16 \\ f((2,2),(0,1)) &= 2\cdot 0 + 2\cdot 2\cdot 1 + 2\cdot 1 = 6 \\ f((0,1),(2,2)) &= 0\cdot 2 + 2\cdot 0\cdot 2 + 1\cdot 2 = 2 \\ f((0,1),(0,1)) &= 0\cdot 0 + 2\cdot 0\cdot 1 + 1\cdot 1 = 1 \end{aligned}$$

so the required matrix is $A = \begin{pmatrix} 16 & 6 \\ 2 & 1 \end{pmatrix}$ [3 marks]

Similarly for the basis (1,1), (0, 1)

$$\begin{aligned} f((1,1),(1,1)) &= 1\cdot 1 + 2\cdot 1\cdot 1 + 1\cdot 1 = 4 \\ f((1,1),(0,1)) &= 1\cdot 0 + 2\cdot 1\cdot 1 + 1\cdot 1 = 3 \\ f((0,1),(1,1)) &= 0\cdot 1 + 2\cdot 0\cdot 1 + 1\cdot 1 = 1 \\ f((0,1),(0,1)) &= 0\cdot 0 + 2\cdot 0\cdot 1 + 1\cdot 1 = 1 \end{aligned}$$

so, in this case, the required matrix is $B = \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix}$. [3 marks]

Also $P = \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}$ so,

$$\begin{aligned} P^TAP &= \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 16 & 6 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 8 & 6 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix} \\ &= B \end{aligned}$$

as required. [4 marks]

A bilnear form is symmetric if $f(u,v) = f(v,u)$. Thus if $v_1, \ldots v_1$ is a basis for V and f is symmetric, $f(v_i, v_j) = f(v_j, v_i)$ for all i, j, so the matrix of f is symmetric.

[2 marks]

The matrix A is orthogonal if $AA^T = I$. If P, Q are orthogonal matrices $PP^T = I$ and $QQ^T = I$. Then

$$PQ(PQ)^T = PQQ^TP^T = PIP^T = I$$

so PQ is orthogonal.

[2 marks]

If now A is the matrix of f with respect to $\{v_1, v_2, \ldots v_n\}$ and P is the change of basis matrix to basis $\{u_1, u_2, \ldots u_n\}$, then the matrix of f with respect to $\{u_1, u_2, \ldots u_n\}$ is P^TAP which has determinant det P^T det A det P. These three determinants are real numbers so det P^T det $A =$ det A det P^T, so the required determinant is equal to det A det P^T det $P =$ det A since P is orthogonal.

[4 marks]

If now A is a symmetric 2×2 which is orthogonal then

$$\begin{pmatrix} a & b \\ b & d \end{pmatrix} \begin{pmatrix} a & b \\ b & d \end{pmatrix} = I_2$$

so $a^2 + b^2 = 1 = b^2 + d^2$ and $b(a + d) = 0$. If $a = -d$, then $det(A) = -1$, so we may suppose that $b = 0$ and $a^2 = b^2 = 1$. Thus the required matrices are I_2 and $-i_2$. [2 marks]

5. The given form is $q(x, y, z) = x^2 + 6xz - 2y^2 + z^2$ so its matrix is

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & 0 \\ 3 & 0 & 1 \end{pmatrix}.$$

[1 mark]

The eigenvalues of A are the zeros of the polynomial

$$\begin{aligned}
&= \det \begin{pmatrix} \lambda - 1 & -0 & -3 \\ 0 & \lambda + 2 & 0 \\ -3 & 0 & \lambda - 1 \end{pmatrix} \\
&= (\lambda - 1)\det \begin{pmatrix} \lambda + 2 & 0 \\ 0 & \lambda - 1 \end{pmatrix} - 3 \det \begin{pmatrix} 0 & \lambda + 2 \\ -3 & 0 \end{pmatrix} \\
&= (\lambda - 1)(\lambda + 2)(\lambda - 1) - 3(3(\lambda + 2)) \\
&= (\lambda + 2)(\lambda - 1)^2 - 9\lambda - 18
\end{aligned}$$

$$
\begin{aligned}
&= (\lambda+2)((\lambda-1)^2-9) \\
&= (\lambda+2)(\lambda^2-2\lambda-8) \\
&= (\lambda+2)(\lambda-4)(\lambda+2).
\end{aligned}
$$

It follows that the eigenvalues are -2 (twice) and 4. [3 marks]

The eigenvectors for eigenvalue -2 are given by

$$
\begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2x \\ -2y \\ -2z \end{pmatrix}
$$

so we obtain the equations $x+3z=-2x$ (or $3x+3z=0$), $-2y=-2y$ (so y is unconstrained) and $3x+z=-2z$ (also giving $x+z=0$). Thus a typical eigenvector is $(x,y,-x)$. [3 marks]

The eigenvectors for eigenvalue 4 are given by

$$
\begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4x \\ 4y \\ 4z \end{pmatrix}.
$$

This time the equations are $x+3z=4x$ (or $x=z$), $-2y=4y$ (giving $y=0$) and $3x+z=4z$ (so $x=z$). A typical eigenvector is of the form $(x,0,x)$. [3 marks]

The required orthogonal matrix P is obtained by putting orthgonormal eigenvectors into columns so

$$
P = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \text{ and } D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{pmatrix}.
$$

[3 marks]

The surface $4X^2-2Y^2-2Z^2=25$ is a hyperboloid of two sheets with circular cross-section, the surface $4X^2-2Y^2-2Z^2=-25$ is a hyperboloid of one sheet while the surface $4X^2-2Y^2-2Z^2=0$ is an elliptic cone.

[5 marks including sketches]

The surface $4X^2-2Y^2-2Z^2=25$ has points arbitarily far from the origin (for example if $Z=0$ we would have $4X^2=25+2Y^2$, where we can obviously find a solution with X as large as we like). Thus this surface is not bounded inside a fixed sphere.

[2 marks]

6. An isometry on $\mathbf{R}^2$ is a map f such that f is a bijection and f preserves distances between points so that for all $u, v \in \mathbf{R}^2$ we have $||u-v|| = ||f(u) - f(v)||$. An example of an isometry which is not a linear map would be (e.g) translation one unit along the direction of the x-axis. Since this does not fix $(0,0)$ it is clearly not a linear map.

[3 marks]

To say that an isometry f is a reflection in a line ℓ means that we obtain the cooordinates of $f(x,y)$ by dropping a perpendicular from (x,y) to a point p on ℓ and extending this perpendicular 'beyond' ℓ for a distance equal to that from (x,y) to p. It is clear from construction that applying f to $f(x,y)$ returns us to (x,y), so f^2 is the identity map.

[3 marks]

The map ϕ will take the vector $(1,0)$ to that obtained by rotating anti-clockwise through 90° so (1,0) maps to (0,1) and $(0,1)$ itself maps to $(-1,0)$. Thus the matrix of ϕ is $M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. [2 marks]

Now consider reflection in the line $y = -x$, the vector $(1,0)$ is sent to $(0,-1)$ and $(0,1)$ is sent to $(-1,0)$, so the matrix A of this reflection is $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$.

[3 marks]

Next consider the matrix B of reflection in the line $y = mx$ where $m =$tan 30°. As always we consider the action of the map on the two unit vectors. A simple diagram, using congruent triangles shows that $(1,0)$ is mapped to the point on the unit cicle making an angle (anti-clockwise) of 60° with the x-axis. This point has coordinates (cos 60°, sin (60°)=$(1/2, \sqrt{3}/2)$. This gives the first column of B. Next consider the map acting on $(0,1)$. We first join the unit vector by a perpendicular to our line (using an angle of 60°). When extending beyond the line, we arrive at a point on the unit circle making an angle of 30° below the x-axis, and so having coordinates (cos 30°, -sin (30°)=$(\sqrt{3}/2, -1/2)$ thus the matrix B is $\begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$.

[5 marks]

The matrix C of the composite map is then

$$AB = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix} = \begin{pmatrix} -\sqrt{3}/2 & 1/2 \\ -1/2 & -\sqrt{3}/2 \end{pmatrix}$$

[2 marks]

The square of C is then

$$\begin{pmatrix} -\sqrt{3}/2 & 1/2 \\ -1/2 & -\sqrt{3}/2 \end{pmatrix} \begin{pmatrix} -\sqrt{3}/2 & 1/2 \\ -1/2 & -\sqrt{3}/2 \end{pmatrix} = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$$

Since this is not the identity matrix, C cannot represent a reflection.

[2 marks]

7. A group is a set G with a law of composition satisfying the following axioms:

(G1) for any $x, y \in G$, xy is in G;

(G2) for any x, y, z in G, $x(yz) = (xy)z$;

(G3) there is an element e in G such that for all $g \in G$, $ge = g = eg$;

(G4) given an element $g \in G$, there is an element g^{-1} of G with $gg^{-1} = 1 = g^{-1}g$.

Given two groups $(G, \circ)$ and $(H, \star)$, a map f is a homomorphism if

$$f(g \circ h) = f(g) \star f(h)$$

for all elements g, h of G

The kernel of f is the set of elements g in G such that $f(g) = e_H$.

The image of f is the set of those elements in h which are images of elements of G under f. [7 marks]

To show that $f(e_G) = e_H$, note that for all $x \in G$ $f(x) = f(e_G x) = f(e_G)f(x)$, so by uniqueness of solutions of equations, $f(e_G) = e_H$. To show that ker f is a subgroup, note that we have already seen that $f(e_G) = e_H$. Also if x, y are in ker f, then $f(x) = e = f(y)$, so $f(xy) = f(x)f(y) = e$ Finally if $g \in$ ker f so that $f(g) = e$, Then $e_H = f(e_G) = f(gg^{-1}) = f(g)f(g^{-1}) = f(g^{-1})$, so g^{-1} is in ker f.

[4 marks]

(a) To check if ϕ is a homomorphism consider two matrices A, B in G, then $\phi(AB)$ is equal to

$$\phi\left(\begin{pmatrix} a_1 & b_1 \\ 0 & b_1 \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ 0 & b_2 \end{pmatrix}\right) = \phi\left(\begin{pmatrix} a_1a_2 & b_2a_1 + a_2b_1 \\ 0 & b_1b_2 \end{pmatrix}\right) = b_2a_1 + a_2b_1.$$

Since this is not equal to $b_1 b_2$ in general, ϕ is not a homomorphism

[2 marks]

(b) Next let A, B be in G and consider

$$\phi(AB) = \phi\left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right) = \phi\left(\begin{pmatrix} 1 & a+b \\ 0 & 1 \end{pmatrix}\right) = a + b.$$

Since the operation in H is addition, this map is also a homomorphism. The kernel is the identity matrix and the image is all of $\mathbf{R}$.

[4 marks]

(c) This map is also a homomorphism since det A det B=det AB. The kernel of this map is the set of matrices of determinant 1 and the image is again all of H. [3 marks]

8. (i) To show e is unique, suppose that G had two identities e_1 and e_2 then $e_1 = g = ge_1$ and $e_2 g = g = ge_2$ for all g in G. Now consider the element $e_1 e_2$. Since e_1 is a left identity, this is e_2, and since e_2 is a right identity this is e_1 so $e_1 = e_2$. [2 marks]

If now an element g of G had two inverses x and y say, we would have

$gx = e = xg$ and $gy = e = yg$. Then $y = (xg)y = x(gy) = xe = x$ using (G2) and the given information. [2 marks]

(ii) Suppose that $a * b = g = a * c$ for some elements a, b, c in G. Multiply the equation $a*b = a*c$ on both sides by the inverse of a to get $a^{-1} * (a*b) = a^{-1} * (a * c)$. Now use associativity to get $(a^{-1} * a) * b = (a^{-1} * a) * c$. Since a^{-1} is the inverse for a, $a^{-1} * a = e$, so we obtain $e \circ b = e \circ c$. The result now follows since e is an identity element. [2 marks]

Now if an element g is repeated in the same row of a table, then g will be of the form $a \circ b$ and also of the form $a \circ c$ for some a, b, and c, so the above argument shows that $b = c$. [1 mark]

For columns, if $a \circ b = c \circ b$, we multiply on right by b^{-1} and again use associativity, inverse and identity to deduce that $a = c$. [2 marks]

(iii) Inspecting the given partial table, we see that $fa = a$ which can only happen in a group when f is the idenity element. This also means that b is the inverse of c (and so c is the inverse of b). Similarly, since d is the inverse of a, a is the inverse of d. We can now fill in more of the partial table:

$\circ$	a	b	c	d	f
a	b	c	?	f	a
b			f	a	b
c		f			c
d	f				d
f	a	b	c	d	f

The entry marked ? cannot be a, b, c or f (already in row) so must be d. Next consider the second entry in the column headed by b. This cannot be c, f or b (all in this column) or a, b, or f (already in row). This entry must also equal d. This gives

$\circ$	a	b	c	d	f
a	b	c	d	f	a
b		d	f	a	b
c		f			c
d	f				d
f	a	b	c	d	f

The remaining entry in the second row must now be c, that in the first column third row must then be d and the missing entry in the second column must be a. We now have

$\circ$	a	b	c	d	f
a	b	c	d	f	a
b	c	d	f	a	b
c	d	f		?	c
d	f	a			d
f	a	b	c	d	f

The entry at ? cannot be a, f or d (already in column) nor d, f, or c (already in row), so must be b the missing entry is that row is then a, that from the same column is c and the final entry b to complete the table as

$\circ$	a	b	c	d	f
a	b	c	d	f	a
b	c	d	f	a	b
c	d	f	a	b	c
d	f	a	b	c	d
f	a	b	c	d	f

[8 marks]

(iv) We are given that $b = a \circ a$. Now $a \circ a \circ a$ is not a (otherwise cancelling a by (ii) would give $b = e$) nor e (otherwise $e \circ a = e$ contrary to definition) so $a \circ a \circ a = c$. We then obtain the table using (i) and (ii))

	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

[3 marks]