Math 744 May 2005

1. (a) Say what it means for $\{v_1, \ldots, v_k\}$ to *span* a vector space V.

Let U be the subspace of $\mathbf{R}^3$ spanned by $u_1 = (1, 0, -1), u_2 = (1, 2, 1)$ and $u_3 = (2, -2, -4)$. Let W be the set of vectors (x, y, z) in $\mathbf{R}^3$ where $x + y + z = 0$. Show that W is a subspace of $\mathbf{R}^3$. Calculate the dimensions of U and of W. Find the subspace $U \cap W$ and determine its dimension. Decide whether or not $\mathbf{R}^3 = U \oplus W$.

(b) Let V be the vector space of polynomials in x of degree at most 3 with coefficients in $\mathbf{R}$. Let the linear map $L : V \to V$ be defined by

$$L(a + bx + cx^2 + dx^3) = b + ax + dx^2 + cx^3$$

Find M, the matrix representing L with respect to the basis $\{1, x, x^2, x^3\}$. What are the eigenvalues and corresponding eigenvectors of M?

2. Define the *kernel*, the *image*, the *rank* and the *nullity* of a linear map. Let $f : V \to W$ be a linear map. Show that the kernel of f is a subspace of W.

Let $f : \mathbf{R}^4 \to \mathbf{R}^4$ be given by

$$f(x, y, z, t) = (x + y + z + t, x + y + 2z + t, 2x + z + 2t, 0).$$

Find a basis for U, the image of f and a basis for W, the kernel of f. Hence compute the rank of f and the nullity of f. Decide whether or not $\mathbf{R}^4$ is $U \oplus W$.

3. Suppose that $\{x_1, x_2, \ldots, x_n\}$ is a basis for a vector space V. Describe the dual space V^* and describe how to define addition and scalar multiplication on V^* [you need not prove that V^* is a vector space]. Define the dual basis $\{\phi_1, \ldots, \phi_n\}$ to $\{x_1, \ldots, x_n\}$ and prove that it is a basis for V^*.

Consider the basis $\{v_1, v_2, v_3\}$ for $\mathbf{R}^3$ where

$$v_1 = (1, 1, 1), \quad v_2 = (1, 2, 4) \quad \text{and } v_3 = (1, -1, 1).$$

Find the dual basis $\{\phi_1, \phi_2, \phi_3\}$ to $\{v_1, v_2, v_3\}$ and find an expression for the value of each of the three maps at a general point of $\mathbf{R}^3$. Hence compute the values of $\phi_1(3, 2, 1)$, $\phi_2(3, 2, 1)$ and $\phi_3(3, 2, 1)$. Now let f be the linear map from $\mathbf{R}^3$ to $\mathbf{R}$ given by $f(x, y, z) = x + 2y + 3z$. Express f as a linear combination of $\{\phi_1, \phi_2, \phi_3\}$.

4. Let f be the bilinear form on $\mathbf{R}^2$ defined by

$$f((x_1, x_2), (y_1, y_2)) = x_1y_1 + 2x_1y_2 + x_2y_2.$$

Let $u_1 = (2, 2), u_2 = (0, 1)$. Compute $f(u_1, u_1), f(u_1, u_2), f(u_2, u_1), f(u_2, u_2)$. Find the matrix A of f relative to the basis $\{u_1, u_2\}$. Find the matrix B of f relative to the basis $\{v_1, v_2\}$, where $v_1 = (1, 1), v_2 = (0, 1)$.

Find the change of basis matrix P from $\{u_1, u_2\}$ to $\{v_1, v_2\}$ and show that $B = P^TAP$.

If f is a symmetric bilinear form, show that the matrix of f is a symmetric matrix.

Define what is meant by saying that a $n \times n$ matrix A is *orthogonal*. If P and Q are orthogonal $n \times n$ matrices, show that PQ is an orthogonal matrix. If A is the matrix of a bilinear form with respect to a basis $v_1, \ldots, v_n$ and the change of basis matrix P to a basis $u_1, \ldots, u_n$ is orthogonal, show that the matrix of f with respect to $u_1, \ldots, u_n$ has the same determinant as that of A.

Is it possible to find a symmetric 2×2 matrix A such that A has determinant 1 and A is orthogonal?

5. Consider the quadratic form

$$q(x, y, z) = x^2 + 6xz - 2y^2 + z^2.$$

Write down the matrix A representing q with respect to the standard basis. Find a diagonal matrix D equivalent to A and an orthogonal matrix P which describes the change of basis from the standard basis to a basis in which q is diagonal. Describe geometrically the surface $q(x, y, z) = 25$. Draw a sketch of the surfaces $q(x, y, z) = 25, q(x, y, z) = -25$ and $q(x, y, z) = 0$. Is it possible to find a sphere of sufficently large radius so that the surface $q(x, y, z) = 25$ lies inside the sphere?

6. Define an *isometry* of $\mathbf{R}^2$. Give an example of an isometry which is not a linear map. Define a *reflection* in a line ℓ and explain why a reflection has square equal to the identity map.

Let ϕ be the linear map which corresponds to rotation of the plane anti-clockwise through an angle of 90° about the origin O. Determine how ϕ maps each of the unit vectors, $(1,0)$ and $(0,1)$. Hence calculate the matrix M of the linear map ϕ.

Let σ_ℓ denote the linear map representing the isometry which is reflection of the plane in the line ℓ with equation $x = -y$, and σ_k correspond to reflection of the plane in the line k with equation $y = mx$, where $m = \tan(30^\circ)$. Given that $\sin(30^\circ) = \cos(60^\circ) = 1/2$ and that $\sin(60^\circ) = \cos(30^\circ) = \sqrt{3}/2$, calculate the matrices A, B of σ_ℓ, σ_k respectively. Compute the matrix C of the composite map $\sigma_\ell\sigma_k$ (where $\sigma_\ell\sigma_k(x) = \sigma_\ell(\sigma_k(x))$) and decide whether this composite map is itself a reflection or not.

7. Define the terms: *group, homomorphism, kernel, image.* Prove that $f(1_G) = 1_H$ and that ker f is a subgroup of G

Decide which of the following maps on the given groups are homomorphisms, finding the kernel and image of those maps which are homomorphisms:

(a) Let G be the group of all 2×2 matrices of the form

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a \in \mathbf{R} \setminus \{0\}, b \in \mathbf{R},$$

under the operation of matrix multiplication. Let H be the group of real numbers, under the operation of addition. Then the map $\phi : G \to H$ is defined by

$$\phi\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = b.$$

(b) Let G be the group of 2×2 matrices of the form $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$: $a \in \mathbf{R}$ under the operation of matrix multiplication. Let H be as in (a) above. Let the map $\phi : G \to H$ be defined by

$$\phi\left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\right) = a.$$

(c) Let G be the group of invertible 2×2 real matrices under matrix multiplication and H be the group of non-zero real numbers under multiplication. The map ϕ is then defined by $\phi(A) =$ det A.

8. (i) Let G be a group with operation $\circ$. Show that the identity element e is unique. Show also that every element of G has a unique inverse.

(ii) Let G be a group with operation $*$ and let a, b, c be elements of G. Show that $a * b = a * c$ implies that $b = c$. Deduce that no element can be repeated in the same row inside a group table. Similarly show that no element can be repeated in the same column of the table.

(iii) The following is a partially completed group table for a group with five elements. Fill in the missing entries. You must justify (entry by entry) why each choice of entry is the only one possible.

$\circ$	a	b	c	d	f
a	b	c			
b			f	a	
c				b	
d	f				
f	a				

(iv) Let G be a group with four distinct elements $\{e, a, b, c\}$, with operation $\circ$ and identity element e. Suppose that $a \circ a$ is not e, but that $a \circ a \circ a \circ a = e$. Given that $a \circ b = c$, determine the composition table for G.