MATH 744 Solutions.

1(a). To say that $\{v_1, v_2, \ldots, v_n\}$ *spans* V means that every element in V can be written as a linear combination

$$\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n.$$

[1 mark]

When we take the vectors $(1,0,-1), (1,-2,1)$ and $(2,2,-4)$, it is clear that the first two are independent, so we investigate what happens if we write

$$(2,2,-4) = \lambda(1,0,-1) + \mu(1,-2,1).$$

This leads to three equations: $2 = \lambda + \mu$, $2 = -2\mu$ and $-4 = -\lambda + \mu$. We see that $\mu = -1$ and so $\lambda = 3$. Since these equations have non-zero solutions, the third vector depends on the first two so U has basis $(1,0,-1)$ and $(1,-2,1)$ and dimension 2. [2 marks]

To show that W is a subspace, note that the zero vector $(0,0,0)$ is in W because $0+0+0=0$. If (x_1,y_1,z_1) and (x_2,y_2,z_2) are in W, so that $x_1+y_1+z_1=0$ and $x_2+y_2+z_2=0$, then

$$(x_1,y_1,z_1) + (x_2,y_2,z_2) = (x_1+x_2, y_1+y_2, z_1+z_2)$$

and since

$$x_1+x_2+y_1+y_2+z_1+z_2 = (x_1+y_1+z_1) + (x_2+y_2+z_3) = 0+0 = 0,$$

it follows that $(x_1+x_2, y_1+y_2, z_1+z_2)$ is in W. Finally, let (x,y,z) be in W and λ be any real number. Then $x+y+z=0$ and $\lambda(x,y,z) = (\lambda x, \lambda y, \lambda z)$. Since

$$\lambda x + \lambda y + \lambda z = \lambda(x+y+z) = \lambda 0 = 0,$$

is follows that $\lambda(x,y,z)$ is also in W and that W is a subspace of V.

[3 marks]

Now

$$\begin{aligned} W &= \{(x,y,z) : x+y+z=0\} \\ &= \{(x,y,z) : z = -(x+y)\} \\ &= \{(x,y,-(x+y))\} \\ &= \{x(1,0,-1) + y(0,1,-1)\} \end{aligned}$$

Since $(1,0,-1)$ and $(0,1,-1)$ are clearly linearly indepedent, they are a basis for W so W also has dimension 2. [2 marks]

Now if (x,y,z) is in $U \cap W$, then $z = -(x+y)$ and so $(x,y,-(x+y))$ is a linear combination of $(1,0,-1)$ and $((1,-2,1)$:

$$(x,y,-(x+y)) = \lambda(1,0,-1) + \mu(1,2,1)$$

this gives $x = \lambda + \mu$, $y = 2\mu$ and $-(x+y) = -\lambda + \mu$. Thus $\mu = y/2$ and $\lambda = x - y/s$ (from the first two equations). The third is then also satisfied, so every element in W is an element of U (also because the 2 basis vectors are in W). It follows that $U = W$, so $U \cap W = U$ and $U + W = U$.

[2 marks]

1 (b). Since $L(1) = x^3 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + 1 \cdot x^3$, the entries in the first column of M are 0, 0, 0, 1. Similarly, we have $L(x) = x^2$, $L(x^2) = x$ and $L(x^3) = 1$. It follows that the matrix M is

$$M = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

[3 marks]

We next compute $\det(\lambda I - M)$ to get

$$\begin{aligned}
\det(\lambda I - M) &= \\
&= \det \begin{pmatrix} \lambda & 0 & 0 & -1 \\ 0 & \lambda & -1 & 0 \\ 0 & -1 & \lambda & 0 \\ -1 & 0 & 0 & \lambda \end{pmatrix} \\
&= \lambda \begin{pmatrix} \lambda & -1 & 0 \\ -1 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} - (-1) \begin{pmatrix} 0 & \lambda & -1 \\ 0 & -1 & \lambda \\ -1 & 0 & 0 \end{pmatrix} \\
&= \lambda(\lambda^3 - (-1)(-\lambda + 0)) + (-\lambda(\lambda) + 1) \\
&= \lambda^4 - \lambda^2 - \lambda^2 + 1 \\
&= (\lambda^2 - 1)^2
\end{aligned}$$

It follows that M has two repeated eigenvalues, namely 1 (twice) and -1 (twice). [3 marks]

When $\lambda = 1$, a vector $v = a + bx + cx^2 + dx^3$ is an eigenvector if $L(v) = v$, so $d + cx + bx^2 + ax^3 = a + bx + cx^2 + dx^3$. This occurs precisely if $d = a$ and $b = c$, so the eigenvectors are the polynomials of the form $a + bx + bx^2 + ax^3$. [2 marks]

When $\lambda = -1$, a vector $v = a + bx + cx^2 + dx^3$ is an eigenvector if $L(v) = -v$, so $d + cx + bx^2 + ax^3 = -a - bx - cx^2 - dx^3$. This occurs precisely if $d = -a$ and $b = -c$, so the eigenvectors are the polynomials of the form $a + bx - bx^2 - ax^3$. [2 marks]

2. The rank of f is the dimension of im f and the nullity of f is the dimension of its kernel. [2 marks]

The matrix of the given linear map is

$$A = \begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & 1 \\ 2 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

To find a basis for the image of f, we need to find a basis for the space spanned by the columns of A: the vectors

$$(1, 1, 2, 0); \quad (1, -1, 0, 0); \quad (-1, 2, 1, 0); \text{ and } (1, 1, 2, 0)$$

Clearly the last equals the first, so the only question is whether the third is a linear combination of the first 2. Consider

$$(-1, 2, 1, 0) = \lambda(1, 1, 2, 0) + \mu(1, -1, 0, 0)$$

This gives $-1 = \lambda + \mu$, $2 = \lambda - \mu$ and $1 = 2\lambda$. Thus $\lambda = 1/2$ and so $\mu = -3/2$. Since these satisfy all three equations, we deduce that $(1, 1, 2, 0)$ and $(1, -1, 0, 0)$ are a basis for the image and the rank of f is 2. [5 marks]

The kernel is the solution set for the equations $Au = 0$, giving

$$x + y - z + t = 0; \quad x - y + 2z + t = 0; \quad 2x + z + 2t = 0.$$

It is clear that if we add the first two equations, we obtain the third, so we are looking for the solution set of

$$x + y - z + t = 0; \quad x - y + 2z + t = 0.$$

The second equation says $y = x + t + 2z$ and then the first becomes $2x + z + 2t = 0$. Thus the solution set consists of vectors of the form

$$(x, x + t - 4(x + t), -2x - 2t, t) = (x, -3x - 3t, -2x - 2t, t).$$

This is clearly a two dimensional space spanned by the vectors $(1,-3,-2,0)$ and $(0,-3,-2,1)$ so the nullity is 2.

[5 marks]

To find a u with $f(u)=0$ and $u=f(v)$, we need u in ker f and in im f. Thus u of the form $\lambda(1,1,2,0)+\mu(1,-1,0,0)=(\lambda+\mu,\lambda-\mu,2\lambda,0)$. Also u is in the kernel of f so

$$\lambda+\mu+\lambda-\mu-2\lambda=0 \text{ and } \lambda+\mu-\lambda+\mu+2(2\lambda)=0$$

thus, we obtain $\lambda=-2\mu$ and so any vector of the form $(\lambda(-1,3,2,0)$ is the required intersection. [2 marks]

Now suppose that $f^2=f$, then if v is in ker f and im f, we see that $v=f(u)$ and $0=f(v)=f(f(u))=f^2(u)=f(u)$ so $v=f(u)=0$ and so the intersection of im f and ker f is zero

[2 marks]

If now v is any element of v, we can write $v=(v-f(v))+f(v))$. Clearly, $f(v)$ is in im f and, when we apply f to $v-f(v)$, we obtain

$$f(v-f(v))=f(v)-f(f(v))=f(v)-f^2(v)=f(v)-f(v)=0$$

so that im f + ker $f=V$ and so the sum is direct.

[4 marks]

3. The dual space is defined to be the set of all linear maps from V to $\mathbf{R}$. Given θ,ϕ in V^*, we can define $\theta+\phi$ by $(\theta+\phi)(x)=\theta(x)+\phi(x)$. Similarly, for λ in $\mathbf{R}$, we define $(\lambda\theta)(x)=\lambda(\theta(x))$.

Given a basis $\{x_1,x_2,\ldots,x_n\}$ for V, we define ϕ_i as the unique linear map which maps x_i to 1, but all other basis elements to 0. To prove this gives a dual basis, suppose first that f is any linear map from V to $\mathbf{R}$. Let λ_j be that scalar which f maps x_j to (so that $\lambda_j=f(x_j)$). Then for any j the map $\lambda_1\phi_1+\cdots\lambda_n\phi_n$ takes x_j to λ_j (since $\phi_i(x_j)=0$ for $i\neq j$). Thus the maps f and $\lambda_1\phi_1+\cdots\lambda_n\phi_n$ agree in their action on a basis for V so must be equal and the vectors $\phi_1,\ldots\phi_n$ span V^*. Now to check linear independence, suppose that $\lambda_1\phi_1+\cdots\lambda_n\phi_n=0$. Then, for any x_j, $(\lambda_1\phi_1+\cdots\lambda_n\phi_n)(x_j)=0$ we also know that $(\lambda_1\phi_1+\cdots\lambda_n\phi_n)(x_j)=\lambda_j$, so each λ_j would then be zero. Thus $\{\phi_1,\ldots,\phi_n\}$ is a basis for V^*. [8 marks]

Thus we have that

$$\begin{array}{lll}\phi_1(v_1)=1; & \phi_1(v_2)=0 & \phi_1(v_3)=0\\ \phi_1(v_2)=0; & \phi_2(v_2)=1 & \phi_2(v_3)=0\\ \phi_1(v_3)=0; & \phi_3(v_2)=0 & \phi_3(v_3)=1.\end{array}$$

[2 marks]

Now if $\phi_1(x,y,z) = a_1x+b_1y+c_1z$, we obtain $a_1+b_1+c_1 = 1$, $a_1+2b_1+4c_1 = 0$ and $a_1 - b_1 + c_1 = 0$. We now solve these equations for a_1, b_1, c_1 to get $2a_1+2c_1 = 1$ (so $c_1 = 1/2-a_1$). We can now re-write the first two to say $b_1 + 1/2 = 1$ (so $b_1 = 1/2$) and $a_1 + 4c_1 = -1$ (so $a_1 = +1$ and $c_1 = -1/2$), so that $\phi_1(x,y,x) = x + y/2 - z/2$. Similar calculations are carried out to determine ϕ_2: we solve

$$a_2 + b_2 + c_2 = 0, \quad a_2 + 2b_2 + 4c_2 = 1, \text{ and } a_2 - b_2 + c_2 = 0$$

These give $a_2 = -1/3, b_2 = 0$ and $c_2 = 1/3$ so that $\phi_2(x,y,x) = -x/3+z/3$. For ϕ_3, we solve

$$a_3 + b_3 + c_3 = 0, \quad a_3 + 2b_3 + 4c_3 = 0, \text{ and } a_3 - b_3 + c_3 = 1.$$

This time the solution is $a_3 = 1/3, b_3 = -1/2$ and $c_3 = 1/6$ so that ϕ_3 is given by $\phi_3(x,y,z) = x/3 - y/2 + z/6$. [5 marks]

Now suppose, that the given f is a linear combination of $\{\phi_1, \phi_2, \phi_3\}$ so that $f = a\phi_1 + b\phi_2 + c\phi_3$. This means that for (x,y,z) in $\mathbf{R}^3$

$$f(x,y,z) = a\phi_1(x,y,z) + b\phi_2(x,y,z) + c\phi_3(x,y,z)$$

Using the expressions for the values of ϕ_1, ϕ_2 and ϕ_3 from earlier in the question, we obtain

$$f(x,y,z) = a(x+y/2-z/2)+b(-x/3+z/3)+c(x/3-y/2+z/6) = x+y+z.$$

Equating coefficients of x gives $1 = a - b/3 + c/3$ (or $3 = 3a - b + c$). The coefficients of y give $1 = a/2 - c/2$ (or $2 = a - c$), and the coefficients of z give $1 = -a/2 + b/3 + c/6$ (or $6 = -3a + 2b + c$). We now need to solve these equations.The second gives $c = a - 2$, so we can re-write the first and third as $3 = 3a - b + (a-2)$ (or $5 = 4a - b$) and $6 = -3a + 2b + (a-2)$ (so that $8 = -2a + 2b$). It only remains to solve the simultaneous equations $5 = 4a - b$ and $8 = -2a + 2b$. Adding twice the first to the second gives $18 = 6a$ (so that $a = 3$ and $c = 1$). Also $b = -5 + 4a = -5 + 12 = 7$.

[5 marks]

4. A map $f : V \times V \to \mathbf{R}$ is a bilinear form if
(BF1) for all u_1, u_2, v in V and $\lambda, \mu \in \mathbf{R}$

$$f(\lambda u_1 + \mu u_2, v) = \lambda(u_1, v) + \mu f(u_2, v),$$

and

(BF2) for all u, v_1, v_2 in V and $\lambda, \mu \in \mathbf{R}$

$$f(u, \lambda v_1 + \mu v_2) = \lambda(u, v_1) + \mu f(u, v_2).$$

[2 marks]

The given map is not bilinear because (for example)

$$f((1,0)+(1,0),(1,0)) = f((2,0),(1,0)) = 4 \neq f((1,0),(1,0))+f((1,0),(1,0))$$

[2 marks]

We are given that $f((x_1, x_2), (y_1, y_2)) = x_1y_1 - x_1y_2 + x_2y_2$. Thus

$$\begin{aligned}
f((2,2),(2,2)) &= 2\cdot 2 - 2\cdot 2 + 2\cdot 2 = 4\\
f((2,2),(0,1)) &= 2\cdot 0 - 2\cdot 1 + 2\cdot 1 = 0\\
f((0,1),(2,2)) &= 0\cdot 2 - 0\cdot 2 + 1\cdot 2 = 2\\
f((0,1),(0,1)) &= 0\cdot 0 - 0\cdot 1 + 1\cdot 1 = 1
\end{aligned}$$

so the required matrix is $A = \begin{pmatrix} 4 & 0 \\ 2 & 1 \end{pmatrix}$. The form is not symmetric.

[4 marks]

Similarly for the basis (1,1), (0, -1)

$$\begin{aligned}
f((1,1),(1,1)) &= 1\cdot 1 - 1\cdot 1 + 1\cdot 1 = 1\\
f((1,1),(0,-1)) &= 1\cdot 0 - 1\cdot(-1) + 1\cdot(-1) = 0\\
f((0,-1),(1,1)) &= 0\cdot 1 - 0\cdot 1 + (-1)\cdot 1 = -1\\
f((0,-1),(0,-1)) &= 0\cdot 0 - 0\cdot -1 + (-1)\cdot(-1) = 1 = 1
\end{aligned}$$

so, in this case, the required matrix is $B = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$. [3 marks]

Also $P = \begin{pmatrix} 1/2 & 0 \\ 0 & -1 \end{pmatrix}$ so,

$$\begin{aligned}
P^TAP &= \begin{pmatrix} 1/2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1/2 & 0 \\ 0 & -1 \end{pmatrix}\\
&= \begin{pmatrix} 1/2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}\\
&= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\\
&= B
\end{aligned}$$

as required. [3 marks]

Now suppose that A is the matrix of a symmetric bilinear form so that $A = A^T$. If B is the matrix of f with respect to another basis and P is the change of basis matrix between these bases, then we know that $B = P^T AP$. Then

$$B^T = (P^T AP)^T = P^T A^T P^{T^T} = P^T AP = B$$

(using that facts that $P^{T^T} = P$ and $A^T = A$). Thus B is also a symmetric matrix. [6 marks]

5. The given form is $q(x, y, z) = x^2 + 6xy + y^2 + 4z^2$ so its matrix is

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

[1 mark]

The eigenvalues of A are the zeros of the polynomial

$$\begin{aligned} \det(\lambda I - A) &= \\ \det \begin{pmatrix} \lambda - 1 & -3 & 0 \\ -3 & \lambda - 1 & 0 \\ 0 & 0 & \lambda - 4 \end{pmatrix} &= \\ (\lambda - 1)\det \begin{pmatrix} \lambda - 1 & 0 \\ 0 & \lambda - 4 \end{pmatrix} + 3 \det \begin{pmatrix} -3 & 0 \\ 0 & \lambda - 4 \end{pmatrix} &= \\ (\lambda - 1)(\lambda - 1)(\lambda - 4) + 3(-3\lambda + 12) &= \\ (\lambda - 4)((\lambda - 1)^2 - 9) &= \\ (\lambda - 4)(\lambda^2 - 2\lambda - 8) &= \\ (\lambda - 4)(\lambda - 4)(\lambda + 2). \end{aligned}$$

It follows that the eigenvalues are 4 (twice) and -2. [4 marks]

The eigenvectors for eigenvalue -2 are given by

$$\begin{pmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2x \\ -2y \\ -2z \end{pmatrix}$$

so we obtain the equations $x + 3y = -2x$ (or $x + y = 0$), $3x + y = -2y$ (also giving $x + y = 0$) and $z = -2z$ (so $z = 0$). Thus a typical eigenvector is $(x, -x, 0)$. [2 marks]

The eigenvectors for eigenvalue 4 are given by

$$\begin{pmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4x \\ 4y \\ 4z \end{pmatrix}$$

This time the equations are $x + 3y = 4x$ (or $x = y$) $3x + y = 4y$ (also giving $x = y$) and $4z = 4z$ (so no constraints on z). A typical eigenvector is of the form $x(1, 1, 0) + z(0, 0, 1)$. [3 marks]

The required P is obtained by putting these eigenvectors into columns so

$$P = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

[2 marks]

The surface becomes $4X^2 + 4Y^2 - 2Z^2 = 25$, a hyperboloid of one sheet with circular cross-sections on planes parallel to the XY-plane (cooling tower shape) [4 marks]

Because of the shape of the surface, it is clear that the points nearest the origin are those on the XY-plane. Since this intersection is a circle of radius 5, the required minimum distance is 5.

[2 marks]

The surface $q(x, y, z) = -25$ will have equation $4X^2 + 4Y^2 - 2Z^2 = -25$ or $-4X^2 - 4Y^2 + 2Z^2 = 25$, so is a hyperboloid of two sheets. [2 marks]

6. An isometry on $\mathbf{R}^2$ is a map from $\mathbf{R}^2$ to itself which is a bijection and preserves distances.

[2 marks]

An example of an isometry which doesn't fix 0 is any translation. Such a map is obviously a distance preserving bijection. [2 marks]

The map ϕ will take the vector $(1, 0)$ to that obtained by rotating anticlockwise through 90° so (1,0) maps to (0,1) and $(0, 1)$ itself maps to $(-1, 0)$. Thus $f(x, y) = (-y, x)$ and the matrix of ϕ is $M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. It is clear that ϕ is a bijection, because ϕ has an inverse. Also ϕ preserves distances because the distance between (x_1, y_1) and (x_2, y_2) is the square root of $(x_1 - x_2)^2 + (y_1 - y_2)^2$, whereas the distance between $\phi(x_1, y_1)$ and $\phi(x_2, y_2)$ is the square root of $(-y_1) - (-y_2))^2 + (x_1 - x_2)^2$. These are clearly equal, so ϕ is an isometry. [6 marks]

Since ℓ is the y-axis, $(1,0)$ is mapped by σ_ℓ to $(-1,0)$, and (0,1) is mapped to itself. It follows that $\sigma_\ell(x,y) = (-x,y)$ and that A, the matrix representing σ_ℓ is $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Clearly σ_ℓ is a bijection (it is self-inverse) and preserves distance: with the notation of first part, we want to compare the square root of $(x_1 - x_2)^2 + (y_1 - y_2)^2$ with square root of $(-x_1 - (-x_2)^2 + (y_1 - y_2)^2$ and these are clearly equal.

[4 marks]

Also k is the line $x = y$, so $(1,0)$ is mapped by σ_ℓ to $(0,1)$, and (0,1) is mapped to (1,0). It follows that B, the matrix representing σ_k is $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

[2 marks]

Finally the composite map will have matrix

$$AB = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

[2 marks]

This is the matrix M and so represents a rotation anti-clockwise through $90°$. The powers of M are $M^2 = -I$, $M^3 = -M$ and $M^4 = I$, so the required integer is 4. This shows that after 4 rotations through $90°$, one returns to the starting position. [2 marks]

7. A group is a set G with a law of composition satisfying the following axioms:

(G1) for any $x, y \in G$, xy is in G;

(G2) for any x, y, z in G, $x(yz) = (xy)z$;

(G3) there is an element e in G such that for all $g \in G$, $ge = g = eg$;

(G4) given an element $g \in G$, there is an element g^{-1} of G with $gg^{-1} = e = g^{-1}g$.

A subgroup of a group G is a non-empty subset of G which it itself a group under the same operation as that of G.

Given two groups $(G, \circ)$ and $(H, \star)$, a map f is a homomorphism if

$$f(g \circ h) = f(g) \star f(h)$$

for all elements g, h of G .

The kernel of f is the set of elements g in G such that $f(g) = e_H$.

The image of f is the set of those elements in h which are images of elements of G under f. [8 marks]

To show that G is a group, first consider the product of two general elements in G:

$$\begin{pmatrix} 1 & a_1 & b_1 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a_2 & b_2 \\ 0 & 1 & c_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a_1 + a_2 & b_1 + c_2 a_1 + b_2 \\ 0 & 1 & c_1 + c_2 \\ 0 & 0 & 1 \end{pmatrix}$$

Since this has the required shape (1's along main diagonal and zero's below it), this checks closure. The multiplication of matrices is always associative, the identity element is in G (with $a = b = c = 0$), so it only remains to see if the inverse of an element of G is also an element of G. By 'inspection'

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -a & ca - b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix}.$$

This checks inverse exist, so G is a group. [4 marks]

Next, to show that we have a subgroup, we consider

$$\begin{pmatrix} 1 & a_1 & b_1 \\ 0 & 1 & a_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a_2 & b_2 \\ 0 & 1 & a_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a_1 + a_2 & b_1 + a_2 a_1 + b{+}_2 \\ 0 & 1 & a_1 + a_2 \\ 0 & 0 & 1 \end{pmatrix}$$

Again this has the shape of a general element in the subset. Again associativity and identity are clear and as for inverses, the above formula shows that the inverse of an element in the subset has the required form.

[3 marks]

To show that ϕ is a homomorphism consider two matrices A, B in G, then

$$\begin{aligned} \phi(AB) &= \phi\left(\begin{pmatrix} 1 & a_1 & b_1 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a_2 & b_2 \\ 0 & 1 & c_2 \\ 0 & 0 & 1 \end{pmatrix}\right) \\ &= \phi\left(\begin{pmatrix} 1 & a_1 + a_2 & b_1 + c_2 a_1 + b{+}_2 \\ 0 & 1 & c_1 + c_2 \\ 0 & 0 & 1 \end{pmatrix}\right) \\ &= a_1 + a_2 \end{aligned}$$

Since $\phi(A) = a_1$ and $\phi(B) = a_2$ and the group operation in H is addition, we see that ϕ is a homomorphism. [2 marks]

The kernel of ϕ is the set of matrices in G with '$a = 0$', so

$$\ker\phi = \{\begin{pmatrix} 1 & 0 & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : b, c \in \mathbf{R}\}.$$

The image of ϕ is the whole of $\mathbf{R}$ since any real number could occur as the appropriate entry of an element A of G. [3 marks]

8. (i) To show e is unique, suppose that G had two identities e_1 and e_2 then $e_1 = g = ge_1$ and $e_2 g = g = ge_2$ for all g in G. Now consider the element $e_1 e_2$. Since e_1 is a left identity, this is e_2, and since e_2 is a right identity this is e_1 so $e_1 = e_2$. [2 marks]

(ii) Suppose that $a \circ b = g = a \circ c$ for some elements a, b, c in G. Multiply the equation $a \circ b = a \circ c$ on both sides by the inverse of a to get $a^{-1} \circ (a \circ b) = a^{-1} \circ (a \circ c)$. Now use associativity to get $(a^{-1} \circ a) \circ b = (a^{-1} \circ a) \circ c$. Since a^{-1} is the inverse for a, $a^{-1} \circ a = e$, so we obtain $e \circ b = e \circ c$. The result now follows since e is an identity element. [2 marks]

Now if an element g is repeated in the same row of a table, then g will be of the form $a \circ b$ and also of the form $a \circ c$ for some a, b, and c, so the above argument shows that $b = c$. [1 mark]

For columns, if $a \circ b = c \circ b$, we multiply on right by b^{-1} and again use associativity, inverse and identity to deduce that $a = c$. [2 marks]

(iii) Inspecting the given partial table, we see that $fd = d$ which can only happen in a group when f is the idenity element. This also means that d is the inverse of a (and so a is the inverse of d). We can now fill in more of the partial table:

$\circ$	a	b	c	d	f
a			?	f	a
b					b
c					c
d	f		b		d
f	a	b	c	d	f

The entry marked ? cannot be a or f (already in row), or b or c (already in column) so must be d. This makes the second entry in this row is not d, f or a and since b is already in this column this entry must be c. The first entry must be b. This gives

$\circ$	a	b	c	d	f
a	b	c	d	f	a
b					b
c					c
d	f		b		d
f	a	b	c	d	f

We can now fill in the fourth row since its second entry must be a and its fourth entry must be c

$\circ$	a	b	c	d	f
a	b	c	d	f	a
b					b
c					c
d	f	a	b	c	d
f	a	b	c	d	f

We can now fill in the first column with third entry d and second entry c. This gives

$\circ$	a	b	c	d	f
a	b	c	d	f	a
b	c			?	b
c	d				c
d	f	a	b	c	d
f	a	b	c	d	f

Since the entry marked ? cannot be b, f,, c or d, it must be a and the corresponding entry in the third row is b. This then forces the table to fill in uniquely as shown

$\circ$	a	b	c	d	f
a	b	c	d	f	a
b	c	d	f	a	b
c	d	f	a	b	c
d	f	a	b	c	d
f	a	b	c	d	f

[5 marks]

(iv) Inspecting the given table, we see that $b \circ (c \circ d) = b \circ a = d$, whereas $(b \circ c) \circ d = a \circ d = b$, so the operation is not associative. [3 marks]

Now suppose G is a group so that we have (from the given information) a partial table

	e	a	b	c	d
e	e	a	b	c	d
a	a	e	c		?
b	b		e		
c	c			e	
d	d				e

If G is to be a group, the entry marked ? cannot be a, e or c (already in row or d (already in column), so must be b. This makes the other missing entry in this row d. Giving

	e	a	b	c	d
e	e	a	b	c	d
a	a	e	c	d	b
b	b		e	?	
c	c			e	
d	d				e

The entry marked now is not c, d or b, e so must be a. This makes the last entry in this row c and the second entry d.

	e	a	b	c	d
e	e	a	b	c	d
a	a	e	c	d	b
b	b	d	e	a	c
c	c			e	?
d	d				e

The missing entry in the last column must be a, the third entry in that row must then be d and the second b. The final row then fills in uniquely and we obtain the table given at the start of the question. Since we have shown that this operation is non-associative, G is not a group

[5 marks].