1. (a) Say what it means for $\{v_1, \ldots, v_k\}$ to *span* a vector space V.

Let U be the subspace of $\mathbf{R}^3$ spanned by $u_1 = (1, 0, -1), u_2 = (1, -2, 1)$ and $u_3 = (2, 2, -4)$. Let W be the set of vectors (x, y, z) in $\mathbf{R}^3$ where $x+y+z=0$. Show that W is a subspace of $\mathbf{R}^3$. Calculate the dimensions of U and of W. Find the subspace $U \cap W$ and determine its dimension. Determine the subspace $U + W$ and decide whether or not $\mathbf{R}^3 = U \oplus W$.

(b) Let V be the vector space of polynomials in x of degree at most 3 with coefficients in $\mathbf{R}$. Let the linear map $L : V \to V$ be defined by

$$L(a + bx + cx^2 + dx^3) = d + cx + bx^2 + ax^3$$

Find M, the matrix representing L with respect to the basis $\{1, x, x^2, x^3\}$. What are the eigenvalues and corresponding eigenvectors of M?

[20 marks]

2. Define the *rank* and *nullity* of a linear map.

Let $f : \mathbf{R}^4 \to \mathbf{R}^4$ given by

$$f(x, y, z, t) = (x + y - z + t, x - y + 2z + t, 2x + z + 2t, 0)$$

Find a basis for the image, U, of f and a basis for the kernel, W, of f. Hence compute the rank of f and the nullity of f. Find a 4-vector u (other than (0,0,0,0)) such that $f(u) = (0, 0, 0, 0)$ and u is of the form $f(v)$ for some 4-vector v.

Now let ϕ be a linear map from vector space V to itself. Suppose that the composite map ϕ^2 is equal to ϕ. Prove that V is the direct sum (im ϕ)$\oplus$ (ker ϕ).

[20 marks]

3. Suppose that $\{x_1, x_2, \ldots, x_n\}$ is a basis for a vector space V. Describe the dual space V^* and describe how to define addition and scalar multiplication on V^* [you need not prove that V^* is a vector space]. Define the dual basis $\{\phi_1, \ldots, \phi_n\}$ to $\{x_1, \ldots, x_n\}$ and prove that it is a basis for V^*.

Consider the basis $\{v_1, v_2, v_3\}$ for $\mathbf{R}^3$ where

$$v_1 = (1, 1, 1), \quad v_2 = (1, 2, 4) \quad \text{and } v_3 = (1, -1, 1).$$

Find the dual basis $\{\phi_1, \phi_2, \phi_3\}$ to $\{v_1, v_2, v_3\}$ and find an expression for the value of each of the three maps at a general point of $\mathbf{R}^3$.

Let f be the linear map from $\mathbf{R}^3$ to $\mathbf{R}$ given by $f(x, y, z) = x + y + z$. Express f as a linear combination of $\{\phi_1, \phi_2, \phi_3\}$.

[20 marks]

4. Define what is meant by saying that f is a **bilinear form** on a vector space V. Explain why the map on $\mathbf{R}^2$ given by $f(x_1, y_1), (x_2, y_2)) = x_1^2$ is not bilinear. Let f be the bilinear form on $\mathbf{R}^2$ defined by

$$f((x_1, x_2), (y_1, y_2)) = x_1 y_1 - x_1 y_2 + x_2 y_2.$$

Let $u_1 = (2, 2), u_2 = (0, 1)$. Compute $f(u_1, u_1), f(u_1, u_2), f(u_2, u_1), f(u_2, u_2)$. Is f a symmetric form? Find the matrix A of f relative to the basis $\{u_1, u_2\}$. Find the matrix B of f relative to the basis $\{v_1, v_2\}$, where $v_1 = (1, 1), v_2 = (0, -1)$.

Find the change of basis matrix P from $\{u_1, u_2\}$ to $\{v_1, v_2\}$ and show that $B = P^T A P$.

Suppose that S is the matrix of a symmetric bilinear form g on V, and let T be the matrix of g with repect to a different basis for V. Is T a symmetric matrix?

[20 marks]

5. Consider the quadratic form

$$q(x, y, z) = x^2 + 6xy + y^2 + 4z^2.$$

Write down the matrix A representing q with respect to the standard basis. Find a diagonal matrix D equivalent to A and an orthogonal matrix P which describes the change of basis from the standard basis to a basis in which q is diagonal. Describe geometrically the surface $q(x, y, z) = 25$. Draw a sketch of the surface. Using this sketch, work out the distance d from the origin of the point on the surface which is closest to the origin. What is the surface $q(x, y, z) = -25$?

[20 marks]

6. Define an *isometry* of $\mathbf{R}^2$. Give an example of an isometry which does not fix the origin, explaining briefly why your chosen map is an isometry.

Let ϕ be the linear map which corresponds to rotation of the plane anti-clockwise through an angle of $90°$ about the origin O. Prove that ϕ is an isometry. Determine how ϕ maps each of the unit vectors, $(1, 0)$ and (0,1). Hence calculate the matrix M of the linear map ϕ. Let σ_ℓ denote the linear map representing the isometry which is reflection of the plane in the line ℓ with equation $x = 0$, and σ_k correspond to reflection of the plane in the line k with equation $x = y$. Explain why each of σ_ℓ and σ_k are isometries. Write down the matrices A, B of σ_ℓ, σ_k respectively. Compute the matrix C of the composite map $\sigma_\ell \sigma_k$ and decide whether this composite map is itself a reflection or not. Find the smallest positive integer such that C^n is the identity matrix, and interpret this geometrically.

[20 marks]

7. Define the terms: *group, subgroup, homomorphism, kernel, image.* Prove that the set G of all 3×3 matrices of the form $\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$, where $a, b, c \in \mathbf{R}$, under the operation of matrix multiplication is a group. Show also that the set of matrices in G of the form $\begin{pmatrix} 1 & a & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}$, with $a, b \in \mathbf{R}$, is a subgroup of G.

Let H be the group of real numbers, under the operation of addition [you need not show that H is a group]. Let $\phi : G \to H$ be defined by

$$\phi\left(\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}\right) = a.$$

Show that ϕ is a homomorphism. Find the kernel and image of ϕ.

[20 marks]

8. (i) Let G be a group. Show that the identity element e is unique.

(ii) Show that, for any $\alpha, \beta, \gamma \in G$, $\alpha * \beta = \alpha * \gamma \Rightarrow \beta = \gamma$. Deduce that no element can be repeated in the same row inside a group table. Similarly show that no element can be repeated in the same column of the table.

(iii) The following is a partially completed group table for a group with five elements. Fill in the missing entries. You must justify (entry by entry) why each choice of entry is the only one possible.

$\circ$	a	b	c	d	f
a					
b					
c					
d	f		b		
f				d	

(iv) Let X be a set with five elements, $\{e, a, b, c, d\}$, with an operation $\circ$ which satisfies the following table:

$\circ$	e	a	b	c	d
e	e	a	b	c	d
a	a	e	c	d	b
b	b	d	e	a	c
c	c	b	d	e	a
d	d	c	a	b	e

Find an example to show that $\circ$ is not an associative operation.

(v) Suppose now that G is a set with five elements $\{E, A, B, C, D\}$ with E being an identity element for G, with the square of each element being E and the elements labelled so that $A \circ B = C$. By considering the possible multiplication table for G, decide whether or not it is possible for G to be a group.

[20 marks]