1.

(a) Let V be a vector space. Say what it means for a subset $\{v_1, \ldots, v_k\}$ of V to be *linearly independent*.

(b) Let $V = \mathbb{R}^3$, and let U be the subspace of V spanned by

$$u_1 = (1, -1, 1), \quad u_2 = (1, 2, -1), \quad u_3 = (3, 0, 1).$$

Find the dimension of U.

(c) Let W be the subspace of V spanned by

$$w_1 = (-4, 1, -2), \quad w_2 = (2, 1, 0), \quad w_3 = (5, 1, 1).$$

Show that $U = W$.

(d) Now let $V = \mathrm{Pol}_3(\mathbb{R})$ be the vector space of polynomials of degree at most three with real coefficients. Let

$$U = \{(2a + b)x^3 + ax^2 - (2a + b)x - b : a, b \in \mathbb{R}\}.$$

Show that U is a subspace of V.

(e) Similarly,

$$W = \{(b - a)x^3 + cx^2 + (a - b)x - c : a, b, c \in \mathbb{R}\}$$

is a subspace of V [you do not need to show this]. What are the dimensions of each of $U, W, U \cap W$ and $U + W$? Is it true or false that $U + W = U \oplus W$?

[20 marks]

2.

(a) Let f be the bilinear form on $\mathbb{R}^2$ defined by

$$f\big((x_1, y_1), (x_2, y_2)\big) = 2x_1x_2 + x_1y_2 + 2y_1y_2.$$

Let $u_1 = (1, -1), u_2 = (1, -2)$. Compute $f(u_1, u_1), f(u_1, u_2), f(u_2, u_1), f(u_2, u_2)$. Find the matrix A of f relative to the basis $\{u_1, u_2\}$. Find the matrix B of f relative to the basis $\{v_1, v_2\}$, where $v_1 = (-2, 1), v_2 = (5, 1)$.

Find the change of basis matrix P from $\{u_1, u_2\}$ to $\{v_1, v_2\}$ and check that $B = P^T AP$.

(b) Consider the quadratic form

$$q(x, y, z) = 4x^2 - 4y^2 + z^2 + 6xy.$$

Give the matrix A representing q with respect to the standard basis. Find a diagonal matrix D equivalent to A and the matrix P which describes the change of basis from the standard basis to the basis in which q is diagonal. What are the rank and signature of q? What type of quadric surface is described by the equation $q(x, y, z) = 2$?

[20 marks]

3. Let V be the vector space of polynomials in x of degree at most 2 with coefficients in $\mathbb{R}$. Let the linear map $\varphi : V \to V$ be defined by

$$\varphi(ax^2 + bx + c) = (3a + b)x^2 + (c + b - 2a)x + 2a + b + c.$$

[You need not show that φ is linear.]

(a) Find M, the matrix representation of φ with respect to the basis $\{x^2, x, 1\}$.

(b) What are the eigenvalues and eigenvectors of M?

(c) Is M diagonalizable?

(d) Find a basis B of V such that the matrix A of φ with respect to B is in Jordan normal form. (When you have found B, you should compute A to check that it is in Jordan normal form.)

[20 marks]

4.

(a) Define the terms: *group, homomorphism, injective, surjective.*

(b) Let G be the group of 2×2 matrices with integer entries, under matrix addition. Let H be the group of integers under addition. [You need not show that these are groups]. Let $\varphi : G \to H$ be defined by

$$\varphi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 3(a+b) - 6(c-d).$$

Show that φ is a homomorphism. State, giving reasons, whether φ is injective. State, giving reasons, whether φ is surjective.

(c) Let G be a group. Which of the following statements are true, and which are false? Give counterexamples to any false statements. You do not need to prove the true statements.

(i) Let $a, b \in G$, and suppose that $ab = a$. Then b is the identity element of G.

(ii) Let $a, b \in G$. Then $ab = ba$.

(iii) If $a, b, c \in G$ with $ab = ac$, then $b = c$.

(iv) If $a, b, c \in G$ with $ab = ca$, then $b = c$.

(d) The following is a partially completed group table. Fill in the missing entries. You must justify why each choice of entry is the only one possible.

*	A	B	C	D	E	F
A	B	?	A	E	?	?
B	C	?	?	?	?	?
C	?	?	?	?	?	?
D	?	E	?	C	?	?
E	?	?	?	A	?	?
F	?	?	?	?	?	C

(e) Name a known group with the same group table.

[20 marks]

5.

(a) Let V and W be finite-dimensional vector spaces, and let $\varphi : V \to W$ be linear. Define the *rank* and *nullity* of φ. State the rank and nullity theorem.

Let $V = \mathbb{R}^3$, and let $W := \mathbb{R}^{2\times 2}$ be the space of real 2×2-matrices. Define $\varphi : V \to W$ by

$$\varphi(x, y, z) = \begin{pmatrix} x+y+z & z+y \\ 2x-y-z & 0 \end{pmatrix}.$$

Show that φ is a linear map and compute its rank and nullity.

(b) Let the linear map $\varphi : \mathbb{R}^4 \to \mathbb{R}^3$ be given by

$$\varphi(x, y, z, w) = (x + z - w, 2x - z - 2w, 4x + z - 4w).$$

[You need not show that φ is linear.]

Find a basis for U, the image of φ, and a basis for W, the kernel of φ. What are the rank and the nullity of φ?

Find bases B of $\mathbb{R}^4$ and C of $\mathbb{R}^3$ such that the matrix A of φ with respect to B and C is in the standard form

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(When you have found B and C, you should compute the matrix A to verify that it has the required form.)

[20 marks]

6. Let V be a vector space.

(a) Say what it means for a function $\varphi : V \to V$ to be an *isomorphism* of V to itself.

(b) Show that the set of isomorphisms from V to itself is a group under composition.

(c) Suppose that V is finite-dimensional, and let $L(V, V)$ be the vector space of linear maps $\varphi : V \to V$, with the usual operations. What is the dimension of $L(V, V)$?

(d) Is the set of isomorphisms $\varphi : V \to V$ a subspace of $L(V, V)$? If so, compute its dimension. Otherwise, show that one of the subspace axioms is not satisfied.

(e) Is $L(V, V)$ a group with respect to composition? (Justify your answer.)

[20 marks]