1. (a) The set $\{v_1, \ldots v_k\}$ spans V if every $v \in V$ can be written as a linear combination $v = \lambda_1 v_1 + \ldots \lambda_k v_k$, for some $\lambda_1, \ldots, \lambda_k \in \mathbf{R}$.

[1 mark]. *Definition from lectures.*

First put u_1, u_2, u_3 as the rows of a matrix, and use row operations to reduce to echelon form:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 2 & 3 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore the space U is spanned by $\{(1,0,2),(0,1,-1)\}$ which are clearly linearly independent and so give a basis for U.

Similarly put w_1, w_2, w_3 as the rows of a matrix, and use row operations to reduce to echelon form:

$$\begin{pmatrix} 1 & 3 & -1 \\ 1 & 4 & -2 \\ 2 & 7 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore the space W also has the same basis as U, namely: $\{(1,0,2),(0,1,-1)\}$, and so $U = W$.

[6 marks]. *Seen similar in exercises.*

(b) In U, taking $a = b = 0$ gives that $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in U$. If $u = \left(\begin{smallmatrix} a & a+b \\ a+b & b \end{smallmatrix}\right) \in U$ and $\lambda \in \mathbf{R}$, then $\lambda u = \lambda\left(\begin{smallmatrix} a & a+b \\ a+b & b \end{smallmatrix}\right) = \left(\begin{smallmatrix} \lambda a & \lambda a+\lambda b \\ \lambda a+\lambda b & \lambda b \end{smallmatrix}\right) \in U$. Also, if $u_1 = \left(\begin{smallmatrix} a_1 & a_1+b_1 \\ a_1+b_1 & b_1 \end{smallmatrix}\right)$ and $u_2 = \left(\begin{smallmatrix} a_2 & a_2+b_2 \\ a_2+b_2 & b_2 \end{smallmatrix}\right)$ are in U then $u_1 + u_2 = \left(\begin{smallmatrix} a_1 & a_1+b_1 \\ a_1+b_1 & b_1 \end{smallmatrix}\right) + \left(\begin{smallmatrix} a_2 & a_2+b_2 \\ a_2+b_2 & b_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a_1+a_2 & a_1+a_2+b_1+b_2 \\ a_1+a_2+b_1+b_2 & b_1+b_2 \end{smallmatrix}\right) \in U$. Hence U is a subspace of V. Proof that W is a subspace of V is almost identical.

[2 marks]. *Standard.*

Typical member of U is $\left(\begin{smallmatrix} a & a+b \\ a+b & b \end{smallmatrix}\right) = a\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right) + b\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, so that $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ span U. Also, $\lambda_1\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right) + \lambda_2\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) \Rightarrow \left(\begin{smallmatrix} \lambda_1 & \lambda_1+\lambda_2 \\ \lambda_1+\lambda_2 & \lambda_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) \Rightarrow \lambda_1 = \lambda_2 = 0$, so that $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ are linearly independent. Hence this gives a basis for U and so U has dimension 2. Similarly, W has basis $\{\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -1 \\ -1 & 1 \end{smallmatrix}\right)\}$ and so W also has dimension 2.

[3 marks]. *Standard.*

For $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ to be in $U \cap W$, we must have $b = c = a + d$ (to be in U) *and* $b = c = a - d$ (to be in W); but $a + d = a - d \iff d = 0$, and so $b = c = a$ and $d = 0$. So, $U \cap W = \{\left(\begin{smallmatrix} a & a \\ a & 0 \end{smallmatrix}\right) : a \in R\}$. Clearly (shown as above) $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is a basis for $U \cap W$ and so $U \cap W$ has dimension 1.

[3 marks]. *Harder, but seen similar.*

Note that $U+W$ is spanned by the union of a basis for U and a basis for W. So, it is spanned by the four vectors: $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, which is a basis for U, and $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}$, which is a basis for W. Then $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ has been repeated twice, and so $U+W$ is spanned by the three vectors: $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}$. These are linearly independent, since $\lambda_1 \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} \lambda_1 & \lambda_1+\lambda_2-\lambda_3 \\ \lambda_1+\lambda_2-\lambda_3 & \lambda_2+\lambda_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \lambda_1 = \lambda_1 + \lambda_2 - \lambda_3 = \lambda_2 + \lambda_3 = 0 \Rightarrow \lambda_1 = \lambda_2 - \lambda_3 = \lambda_2 + \lambda_3 = 0 \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0$. Hence these three vectors form a basis for $U+W$, giving that $U+W$ has dimension 3.

[3 marks]. *Harder. Unseen.*

Finally note that, since $\dim(U \cap W) = 1$, we do *not* have $U \cap W = \{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\}$, and so $U + W = U \oplus W$ (note that the definition of $S = U \oplus W$ is that *both* $S = U + W$ *and* $U \cap W = \{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\}$).

[2 marks]. *Seen similar in exercises (once).*

20 marks in total for Question 1

2. (a) Let $e_1 = 1, e_2 = x, e_3 = x^2$. Then $L(e_1) = L(1) = 1 = 1 \cdot e_1 + 0 \cdot e_2 + 0 \cdot e_3$, so that the first column of the matrix should have entries $1, 0, 0$. Similarly, $L(e_2) = 0 \cdot e_1 + 1 \cdot e_2 + 1 \cdot e_3$ and $L(e_3) = 0 \cdot e_1 + 1 \cdot e_2 + 1 \cdot e_3$, so that the matrix is:

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

[2 marks]

If we now compute $\det(\lambda I - M) = (\lambda - 1)((\lambda - 1)^2 - 1) = \lambda(\lambda - 1)(\lambda - 2)$, we see that the possible eigenvalues are $\lambda = 0, 1, 2$.

When $\lambda = 0$, a vector $v = a + bx + cx^2$ is an eigenvector with eigenvalue 0 iff $L(v) = 0 \cdot v$ iff $a + (b + c)x + (b + c)x^2 = 0$ iff $a = 0$ and $b + c = 0$ iff $a = 0$ and $c = -b$ iff v is of the form $bx - bx^2$ $(b \neq 0)$.

When $\lambda = 1$, a vector $v = a + bx + cx^2$ is an eigenvector with eigenvalue 1 iff $L(v) = 1 \cdot v$ iff $a + (b + c)x + (b + c)x^2 = a + bx + cx^2$ iff $b + c = b$ and $b + c = c$ iff $b = c = 0$ iff v is of the form a $(a \neq 0)$.

When $\lambda = 2$, a vector $v = a + bx + cx^2$ is an eigenvector with eigenvalue 2 iff $L(v) = 2 \cdot v$ iff $a + (b + c)x + (b + c)x^2 = 2a + 2bx + 2cx^2$ iff $a = 2a$ and $b + c = 2b$ and $b + c = 2c$ iff $a = 0$ and $c = b$ iff v is of the form $bx + bx^2$ $(b \neq 0)$.

[5 marks] *Seen similar in exercises.*

(b) (i) The *rank* of ϕ is the dimension of the image of ϕ. The *nullity* of ϕ is the dimension of the kernel of ϕ. That rank & nullity theorem states that rank(ϕ) + nullity(ϕ) = dim(V).

[2 marks] *From lectures.*

(ii) Let B be the matrix of F wrt the basis E_1, E_2, E_3, E_4. We have $F(E_1) = \left(\begin{smallmatrix} 1 & 0 \\ 2 & 0 \end{smallmatrix}\right) = 1 \cdot E_1 + 0 \cdot E_2 + 2 \cdot E_3 + 0 \cdot E_4$, so that the entries of the first column of B are $\begin{smallmatrix} 1 \\ 0 \\ 2 \\ 0 \end{smallmatrix}$. Similarly, we have $F(E_2) = \left(\begin{smallmatrix} 0 & 2 \\ 0 & 2 \end{smallmatrix}\right) = 0 \cdot E_1 + 2 \cdot E_2 + 0 \cdot E_3 + 2 \cdot E_4$, which gives the entries of the second column of B. Similarly $F(E_3) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) = 1 \cdot E_1 + 0 \cdot E_2 + 0 \cdot E_3 + 0 \cdot E_4$, which gives the entries of the third column of B. Finally, $F(E_4) = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right) = 0 \cdot E_1 + 1 \cdot E_2 + 0 \cdot E_3 + 1 \cdot E_4$, which gives the entries of the fourth column of B. So, B is: $\left(\begin{smallmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \end{smallmatrix}\right)$.

[3 marks]. *Seen similar in exercises.*

Applying column operations to B as follows: $C_3 \to C_3 - C_1$, then $C_2 \to (1/2)C_2$, then $C_4 \to C_4 - C_2$, and then $C_3 \to (-1/2)C_3$, gives the matrix $\left(\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{smallmatrix}\right)$, which is in column echelon form. The first three columns of B give a basis for the image of F, that is, a basis for the image of F is: $1 \cdot E_1 + 0 \cdot E_2 + 2 \cdot E_3 + 0 \cdot E_4$, $0 \cdot E_1 + 1 \cdot E_2 + 0 \cdot E_3 + 1 \cdot E_4$ and $0 \cdot E_1 + 0 \cdot E_2 + 1 \cdot E_3 + 0 \cdot E_4$, that is to say, a basis for the image of F is: $\{\left(\begin{smallmatrix} 1 & 0 \\ 2 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)\}$. [*Alternative Method:* we could have found a basis for the image of F directly from the definition of F (without needing B) by observing that $F(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) = \left(\begin{smallmatrix} a+c & 2b+d \\ 2a & 2b+d \end{smallmatrix}\right) = a\left(\begin{smallmatrix} 1 & 0 \\ 2 & 0 \end{smallmatrix}\right) + (b+2d)\left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right) + c\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, so that $\left(\begin{smallmatrix} 1 & 0 \\ 2 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ span the image of F, and are clearly linearly independent, and so give a basis for the image of F].

[3 marks]. *Unseen.*

Solving for $B\left(\begin{smallmatrix} a \\ b \\ c \\ d \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}\right)$, we first apply row operations to B as follows: $R_3 \to R_3 - 2R_1$ and $R_4 \to R_4 - R_2$ gives the row echelon form matrix: $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. This gives only two independent equations: $a + c = 0$, $2b + d = 0$ and $-2c = 0$, equivalent to $a = c = 0$ and $d = -2b$, so that the general solution for a, b, c, d is: $0, b, 0, -2b$, that is: $0 \cdot E_1 + bE_2 + 0 \cdot E_3 - 2bE_4$. The typical member of the kernel of F is then: $\left(\begin{smallmatrix} 0 & b \\ 0 & -2b \end{smallmatrix}\right) = b\left(\begin{smallmatrix} 0 & 1 \\ 0 & -2 \end{smallmatrix}\right)$. So, $\left(\begin{smallmatrix} 0 & 1 \\ 0 & -2 \end{smallmatrix}\right)$ spans the kernel of F and is clearly linearly independent. So, $\left(\begin{smallmatrix} 0 & 1 \\ 0 & -2 \end{smallmatrix}\right)$ is a basis for the kernel of F. [*Alternative Method:* we could have found a basis for the kernel of F directly from the definition of F (without needing B) by observing that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \ker F \iff F(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) \iff \left(\begin{smallmatrix} a+c & 2b+d \\ 2a & 2b+d \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) \iff$ a + c = 0, 2a = 0, 2b+d = 0 $\iff$ a=c=0, d=-2b $\iff \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & b \\ 0 & -2b \end{smallmatrix}\right) = b\left(\begin{smallmatrix} 0 & 1 \\ 0 & -2 \end{smallmatrix}\right)$, giving $\left(\begin{smallmatrix} 0 & 1 \\ 0 & -2 \end{smallmatrix}\right)$ as a basis for the kernel of F.]

Since a basis for the image of F has three elements, it follows that $\text{rank}(F) = 3$. Since a basis for the kernel of F has one element, it follows that $\text{nullity}(F) = 1$. Also, $\dim(V) = 4$, since $\{E_1, E_2, E_3, E_4\}$ is a basis for V. So, the rank & nullity theorem is verified in this case as: $3 + 1 = 4$.

[5 marks]. *Seen (somewhat) similar in exercises.*

20 marks in total for Question 2

3. (a) A *group* is a set G together with a binary operation $*$ such that: **(1)** for all $g_1, g_2 \in G$, $g_1 * g_2 \in G$; **(2)** for all $g_1, g_2, g_3 \in G$, $g_1 * (g_2 * g_3) = (g_1 * g_2) * g_3$; **(3)** there exists an element $e \in G$ such that, for all $g \in G$, $e * g = g * e = g$; **(4)** for every $g \in G$, there exists $g^{-1} \in G$ such that $g * g^{-1} = g^{-1} * g = e$. If G, H are groups, then a map $\phi : G \to H$ is a *homomorphism* if, for all $g_1, g_2 \in G$, $\phi(g_1 *_1 g_2) = \phi(g_1) *_2 \phi(g_2)$, where $*_1$ is the group law in G and $*_2$ is the group law in H. The map ϕ is *injective* if, for all $g_1, g_2 \in G$, $\phi(g_1) = \phi(g_2) \Rightarrow g_1 = g_2$. The map ϕ is *surjective* if, for all $h \in H$, there exists $g \in G$ such that $\phi(g) = h$.

[4 marks]. *Standard definitions from lectures.*

For any $g_1, g_2 \in G$ we have
$\phi(g_1 + g_2) = \begin{pmatrix} 2(g_1+g_2) & g_1+g_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2g_1 & g_1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 2g_2 & g_2 \\ 0 & 0 \end{pmatrix} = \phi(g_1) + \phi(g_2)$.
Hence ϕ is a homomorphism.

For any $g_1, g_2 \in G$, $\phi(g_1) = \phi(g_2) \Rightarrow \begin{pmatrix} 2g_1 & g_1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2g_2 & g_2 \\ 0 & 0 \end{pmatrix} \Rightarrow g_1 = g_2$, so that ϕ is injective.

The element $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in H$ does not occur as $\phi(g)$ for any $g \in G$ (since $\phi(g)$ always has 00 as its bottom row), so that ϕ is not surjective.

[3 marks]. *Seen somewhat similar in exercises.*

(b) (i) Suppose that e_1 and e_2 were both (2-sided) identity elements. Then $e_1 * e_2 = e_1$, since e_2 is an identity. Similarly, $e_1 * e_2 = e_2$. Hence $e_1 = e_2$.

[1 marks]. *Seen in lectures.*

Let $\alpha * \beta = e$. Let δ be the (2-sided) inverse of α, and multiply both sides of the equation on the left by δ. Then $\delta * (\alpha * \beta) = \delta * e = \delta$ (since e is identity), so that $(\delta * \alpha) * \beta = \delta$ (assoc.) and so $\beta = \delta$. Now multiply both sides on the right by α, giving $\beta * \alpha = \delta * \alpha = e$.

[2 marks]. *Unseen*

(ii) Suppose $\alpha * \beta = \alpha * \gamma$. Multiply both sides on the left by δ, the inverse of α. Then $\delta * (\alpha * \beta) = \delta * (\alpha * \gamma)$, giving $(\delta * \alpha) * \beta = (\delta * \alpha) * \gamma$ [by associativity], and so $e * \beta = e * \gamma$, finally giving: $\beta = \gamma$, as required. The values of $\alpha * g$, as g runs through all the members of the group give the 'α' row of the group table; if two of these were the same, we would have $\alpha * \beta = \alpha * \gamma$ for distinct $\beta \neq \gamma$, contradicting the previous result. Similarly, $\beta * \alpha = \gamma * \alpha \Rightarrow \beta = \gamma$ gives that no element can be repeated in the same column.

[3 marks]. *Seen on exercise sheet.*

(iii) From the already provided entry $E * F = E$, we deduce (after multiplying both sides on left by the inverse of E) that F is the identity element. This allows us to fill in the bottom row as ABCDEF and similarly the right hand column. Having done this, we use the given entry $B * A = F$, the identity element, and the second part of (i), to deduce that $A * B = F$. At this point we have:

*	A	B	C	D	E	F
A	D	F	?	C	?	A
B	F	?	?	?	?	B
C	?	?	?	?	?	C
D	B	?	A	E	?	D
E	?	A	B	?	?	E
F	A	B	C	D	E	F

From now on, we can fill in all the remaining entries by using only the 'no-element-repeated-in-the-same-row-or-column' rule. For example, this forces $B*D$ to be A. The following gives a possible order in which the remaining 16 entries can be fixed using this rule.

*	A	B	C	D	E	F
A	D	F	3	C	2	A
B	F	6	4	1	5	B
C	9	7	14	15	13	C
D	B	8	A	E	11	D
E	10	A	B	16	12	E
F	A	B	C	D	E	F

The final table must then be

*	A	B	C	D	E	F
A	D	F	E	C	B	A
B	F	E	D	A	C	B
C	E	D	F	B	A	C
D	B	C	A	E	F	D
E	C	A	B	F	D	E
F	A	B	C	D	E	F

[5 marks]. *Seen similar on Ex Sheet (but this one is harder).*

Finally, note that then $(A*A)*A = D*A = B$, but $A*(A*A) = A*D = C$, violating associativity. Since the above is the unique way of completing the table in a way compatible with (i),(ii), and since any group (by definition) satisfies associativity, there is no way of completing the given table to form a group table.

[2 marks]. *Unseen.*

20 marks in total for Question 3

4. (a) (i) First note that $\sigma_\ell, \sigma_m, \rho_{A,2\alpha}$ all leave A unchanged, so that $\sigma_m\sigma_\ell(A) = A = \rho_{A,2\alpha}(A)$. Now, let B be any point on ℓ distinct from A, let $B' = \sigma_m(B)$ and let n be the line through A and B'. Let the point Q be the intersection of m and the line BB'. Now, $|AQ| = |AQ|$ and $|BQ| = |B'Q|$ and angle AQB equals angle AQB' equals $\pi/2$. So, triangle AQB is congruent to AQB', giving that $|AB| = |AB'|$ and angle QAB' is the same as angle BAQ, namely: α. It follows that $B' = \rho_{A,2\alpha}(B)$. Further, $\sigma_\ell(B) = B$, since B lies on ℓ. So, we've shown that $\sigma_m\sigma_\ell(B) = B' = \rho_{A,2\alpha}(B)$. Similarly, let k be the line through A at angle $-\alpha$ from ℓ, and let C be any point on k distinct from A. By a simlar argument to above, $\sigma_m\sigma_\ell(C) = \rho_{A,2\alpha}(C)$. This shows that $\sigma_m\sigma_\ell$ and $\rho_{A,2\alpha}$ agree on the three non-collinear points A, B, C. Since these are isometries, and since any isometry is determined by its effect on 3 non-collinear points, we conclude that $\sigma_m\sigma_\ell = \rho_{A,2\alpha}$, as required [it helps also to draw a quick diagram of the above].

[**6 marks**]. *Bookwork from lectures.*

(ii) Let r be the line through B at angle $-\beta/2$ from s. By part (i), we have: $\sigma_s\sigma_r = \rho_{B,2(\beta/2)} = \rho_{B,\beta}$. Similarly, let t be the line through B at angle $\beta/2$ from s. By part (i), we have: $\sigma_t\sigma_s = \rho_{B,2(\beta/2)} = \rho_{B,\beta}$. So, $\rho_{B,\beta}\sigma_s = \sigma_s\rho_{B,\beta} \iff (\sigma_t\sigma_s)\sigma_s = \sigma_s(\sigma_s\sigma_r) \iff \sigma_t(\sigma_s\sigma_s) = (\sigma_s\sigma_s)\sigma_r \iff \sigma_t = \sigma_r \iff t = r \iff$ the angle between r and t is 0 or $\pi \iff \beta/2 + \beta/2 = 0$ or π [since the angle from r to t is the "angle from r to s plus angle from s to t] $\iff \beta = 0$ or π, as required.

[**5 marks**]. *Seen similar in exercises.*

(b) A matrix M is *orthogonal* if $MM^T = I$. Let $P = \left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right)$ and $Q = \left(\begin{smallmatrix} a_2 & b_2 \\ c_2 & d_2 \end{smallmatrix}\right)$. Then

$$(PQ)^T = \begin{pmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{pmatrix}^T = \begin{pmatrix} a_1a_2 + b_1c_2 & c_1a_2 + d_1c_2 \\ a_1b_2 + b_1d_2 & c_1b_2 + d_1d_2 \end{pmatrix}$$

$$= \begin{pmatrix} a_2 & c_2 \\ b_2 & d_2 \end{pmatrix}\begin{pmatrix} a_1 & c_1 \\ b_1 & d_1 \end{pmatrix} = Q^TP^T.$$

[**4 marks**]

I is orthogonal, since $II^T = I$. If P, Q are orthogonal then $PP^T = I$ and $QQ^T = I$, so that $(PQ)(PQ)^T = (PQ)Q^TP^T = P(QQ^T)P^T = PIP^T = PP^T = I$, so that PQ is also orthogonal. Also, if P is orthogonal, then $P^T = P^{-1}$, so that $P^{-1}(P^{-1})^T = P^{-1}(P^T)^T = P^{-1}P = I$, so that P^{-1} is also orthogonal. Hence, the set of orthogonal 2×2 matrices contains the identity, is closed, contains inverses, and is associative (since matrix multiplication is always associative), and so is a group.

[**5 marks**]. *Seen on exercise sheet.*

20 marks in total for Question 4

5. (a) We compute: $f(u_1,u_1) = 1\cdot 1 + (-1)\cdot 1 + 2\cdot(-1)\cdot(-1) = 2$, $f(u_1,u_2) = 1\cdot 1 + (-1)\cdot 1 + 2\cdot(-1)\cdot 2 = -4$, $f(u_2,u_1) = 1\cdot 1 + 2\cdot 1 + 2\cdot 2\cdot(-1) = -1$, $f(u_2,u_2) = 1\cdot 1 + 2\cdot 1 + 2\cdot 2\cdot 2 = 11$. So, the matrix of f wrt u_1, u_2 is $A = \left(\begin{smallmatrix} 2 & -4 \\ -1 & 11 \end{smallmatrix}\right)$.

[2 marks]

Similarly, $f(v_1,v_1) = 2\cdot 2 + 1\cdot 2 + 2\cdot 1\cdot 1 = 8$, $f(v_1,v_2) = 2\cdot 0 + 1\cdot 0 + 2\cdot 1\cdot 3 = 6$, $f(v_2,v_1) = 0\cdot 2 + 3\cdot 2 + 2\cdot 3\cdot 1 = 12$, $f(v_2,v_2) = 0\cdot 0 + 3\cdot 0 + 2\cdot 3\cdot 3 = 18$. So, the matrix of f wrt v_1, v_2 is $B = \left(\begin{smallmatrix} 8 & 6 \\ 12 & 18 \end{smallmatrix}\right)$.

[2 marks]

Now, note that $v_1 = 1\cdot u_1 + 1\cdot u_2$, so that "1" and "1" are the entries of the first column of the change-of-basis matrix. Similarly, $v_2 = (-1)\cdot u_1 + 1\cdot u_2$, so that "-1" and "1" are the entries of the second column of the change-of-basis matrix. This gives $P = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix}\right)$ as the required change-of-basis matrix. Finally, check that: $P^TAP = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix}\right)^T \left(\begin{smallmatrix} 2 & -4 \\ -1 & 11 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 2 & -4 \\ -1 & 11 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 7 \\ -3 & 15 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 8 & 6 \\ 12 & 18 \end{smallmatrix}\right) = B$, as required.

[3 marks]. *Whole of Part (a): seen similar (once) in exercises.*

(b) We take A, the matrix representing the quadratic form $q(x,y,z)$, form $(A|I)$, and then use row & column opertations $R_2 \to R_2 + R_1$ & $C_2 \to C_2 + C_1$ followed by: $R_3 \to R_3 - (1/2)R_2$ $C_3 \to C_3 - (1/2)C_2$, with only the column operations being performed on I, as follows:

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ -1 & 3 & 1 & 0 & 1 & 0 \\ 0 & 1 & 4 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 4 & 0 & 0 & 1 \end{array}\right)$$
$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -\frac{1}{2} \\ 0 & 2 & 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & \frac{7}{2} & 0 & 0 & 1 \end{array}\right).$$

[6 marks].

Now let: $A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 3 & 1 \\ 0 & 1 & 4 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{5}{2} \end{pmatrix}$, $P = \begin{pmatrix} 1 & 1 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}$,

$$Q = P^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix},$$

then $D = P^TAP$ and $A = Q^TDQ$. Here, A represents the quadratic form wrt x, y, z and D represents it wrt new variables r, s, t given by $\left(\begin{smallmatrix} r \\ s \\ t \end{smallmatrix}\right) = Q \left(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right)$, that is: $r = x - y$, $s = y + z/2$, $t = z$.

[3 marks]

The rank of q is 3 (which is the number of nonzero entries of D), and the signature of q is the number of positive entries of D minus the number of negative entries $= 3 - 0 = 3$. The surface $q(x, y, z) = 2$ becomes $r^2 + 2s^2 + (7/2)t^2 = 2$, in r, s, t coordinates, which is an ellipsoid. The sketch should look identical to the standard sketch of an ellipsoid, except that the x, y, z axes should be labelled r, s, t (if drawn it wrt r, s, t). [If drawn wrt x, y, z then it should be made clear in the diagram that the axes of the surface are: $y = z = 0$, $x - y = z = 0$, $x - y = y + z/2 = 0$].

[4 marks]. *Whole of Part (b): seen similar in exercises.*

20 marks in total for Question 5

6. (a) We have, using the addition formula for sin and cos:

$$\begin{aligned} A(\theta_1+\theta_2) &= \begin{pmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{pmatrix} \\ &= \begin{pmatrix} \cos(\theta_1)\cos(\theta_2)-\sin(\theta_1)\sin(\theta_2) & -\sin(\theta_1)\cos(\theta_2)-\cos(\theta_1)\sin(\theta_2) \\ \sin(\theta_1)\cos(\theta_2)+\cos(\theta_1)\sin(\theta_2) & \cos(\theta_1)\cos(\theta_2)-\sin(\theta_1)\sin(\theta_2) \end{pmatrix} \\ &= \begin{pmatrix} \cos(\theta_1) & -\sin(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) \end{pmatrix} \begin{pmatrix} \cos(\theta_2) & -\sin(\theta_2) \\ \sin(\theta_2) & \cos(\theta_2) \end{pmatrix} = A(\theta_1)A(\theta_2). \end{aligned}$$

[2 marks].

Letting $\theta_1 = \theta$ and $\theta_2 = -\theta$ gives $A(\theta)A(-\theta) = A(\theta-\theta) = A(0) = I$, so that $A(\theta)^{-1} = A(-\theta)$.

[1 mark].

λ is an eigenvalue of $A(\theta)$ when $\det(\lambda I - A(\theta)) = 0$, that is, when λ is a root of the equation $(\lambda - \cos(\theta))^2 + \sin^2(\theta) = 0$.

That is: $\lambda^2 - 2\cos(\theta)\lambda + 1 = 0$, which has a real root iff the discriminant $4(\cos^2(\theta)-1) \geq 0$; but this is never true for $0 < \theta < \pi$ since then $-1 < \cos(\theta) < 1$ and $\cos^2(\theta) - 1 < 0$.

[4 marks]. *Whole of Part (a): seen similar in exercises.*

(b) The dual space V^* is defined to be the set of all linear maps from V to $\mathbf{R}$. Given $\theta, \phi \in V^*$, we can define $\theta + \phi$ by: $(\theta+\phi)(x) = \theta(x) + \phi(x)$, for all $x \in V$. Similarly, for $\lambda \in \mathbf{R}$, define $\lambda\theta$ by $(\lambda\theta)(x) = \lambda(\theta(x))$, for all $x \in V$. Given a basis $\{x_1, \ldots, x_n\}$ for V, the i-th member of the dual basis, ϕ_i, is defined to be the unique linear map from V to $\mathbf{R}$ such that $\phi_i(x_i) = 1$ and $\phi_i(x_j) = 0$, for all $j \neq i$. Suppose $f \in V^*$; define $\lambda_j = f(x_j)$ for all j; then $(\lambda_1\phi_1 + \ldots + \lambda_n\phi_n)(x_j) = \lambda_j \cdot \phi_j(x_j)$ [since $\phi_i(x_j) = 0$, for all $j \neq i$] $= \lambda_j$ [since $\phi_j(x_j) = 1$]. Hence, f and $\lambda_1\phi_1 + \ldots + \lambda_n\phi_n$ both take the same values on each of $x_1, \ldots x_n$, giving that $f = \lambda_1\phi_1 + \ldots + \lambda_n\phi_n$ [since any linear map is completely determined by its values on a basis]. Hence, $\{\phi_1, \ldots \phi_n\}$ spans V^*. Now suppose that $\lambda_1\phi_1 + \ldots + \lambda_n\phi_n = 0$ for some $\lambda_1, \ldots, \lambda_n$. Then, for any j, $(\lambda_1\phi_1 + \ldots + \lambda_n\phi_n)(x_j) = 0$, and so $\lambda_j \cdot 1 = 0$; hence $\lambda_1 = \ldots = \lambda_n = 0$, and so $\phi_1, \ldots \phi_n$ are linearly independent. Hence $\{\phi_1, \ldots \phi_n\}$ is a basis for V^*.

[8 marks] *Bookwork*

The matrix whose rows are v_1, v_2, v_3 has determinant, expanding along the first row: $1(0+2) - 1(-6-0) + 2(4-0) = 16 \neq 0$, and so the the vectors must form a basis of $\mathbf{R}^3$ (since the determinant is nonzero). We first wish to compute $\phi_1((x,y,z)) = a_1x + b_1y + c_1z \in V^*$, which satisfies $\phi_1(v_1) = 1, \phi_1(v_2) = 0, \phi_1(v_3) = 0$. This gives the three equations: $a_1 + b_1 + 2c_1 = 1, 2a_1 - c_1 = 0, 2b_1 - 3c_1 = 0$, giving: $a_1 = 1/8, b_1 = 3/8, c_1 = 1/4$, so that: $\phi_1(x,y,z) = \frac{1}{8}x + \frac{3}{8}y + \frac{1}{4}z$.

Similarly, $\phi_2((x,y,z)) = a_2x+b_2y+c_2z$ satisfies $\phi_2(v_1)=0, \phi_2(v_2)=1, \phi_2(v_3)=0$. This gives the three equations: $a_2+b_2+2c_2=0, 2a_2-c_2=1, 2b_2-3c_2=0$, giving: $a_2=\frac{7}{16}, b_2=-\frac{3}{16}, c_2=-\frac{1}{8}$, so that: $\phi_2(x,y,z)=\frac{7}{16}x-\frac{3}{16}y-\frac{1}{8}z$. Finally, $\phi_3((x,y,z)) = a_3x+b_3y+c_3z$ satisfies $\phi_3(v_1)=0, \phi_3(v_2)=0, \phi_3(v_3)=1$. This gives the three equations: $a_3+b_3+2c_3=0, 2a_3-c_3=0, 2b_3-3c_3=1$, giving: $a_3=-\frac{1}{16}, b_3=\frac{5}{16}, c_3=-\frac{1}{8}$, so that: $\phi_3(x,y,z)=-\frac{1}{16}x+\frac{5}{16}y-\frac{1}{8}z$.

We can now compute: $\phi_1((-2,1,1))=\frac{1}{8}(-2)+\frac{3}{8}(1)+\frac{1}{4}(1)=\frac{3}{8}$, $\phi_2((-2,1,1))=\frac{7}{16}(-2)-\frac{3}{16}(1)-\frac{1}{8}(1)=-\frac{19}{16}$, and $\phi_3((-2,1,1))=-\frac{1}{16}(-2)+\frac{5}{16}(1)-\frac{1}{8}(1)=\frac{5}{16}$.

[5 marks] *Seen similar in exercises (but this one is slightly harder).*