THE UNIVERSITY
of LIVERPOOL

1. (a) (*i*) Let $f(x+iy) = u(x,y)+iv(x,y)$, where x, y, u and v are real. Derive the Cauchy-Riemann equations which hold where f is complex differentiable.

(*ii*) Find the real and imaginary parts of the function

$$f(z) = i|z|^2 - z^2 .$$

Show that f satisfies both Cauchy-Riemann equations if and only if $z = 0$.

(*iii*) Find a holomorphic function on $\mathbf{C}$ with the real part $u(x,y) = y - 2xy$.

(b) Sketch the path $\gamma : [-1,1] \to \mathbf{C}$ given by

$$\gamma(t) = \begin{cases} 2t\,, & -1 \le t \le 0\,, \\ -t - it\,, & 0 \le t \le 1\,. \end{cases}$$

Evaluate $\int_\gamma \operatorname{Im} z\,dz$ and $\int_\gamma z\,dz$.

2. (a) State Cauchy's Integral Formula.

(b) Evaluate the following integrals, in each case giving a brief justification for your answer:

$$\int_{\gamma(i;1)} \frac{dz}{z^2-4}\,; \qquad \int_{\gamma(-6;5)} \frac{dz}{z^2-4}\,; \qquad \int_{\gamma(-i;10)} \frac{dz}{z^2-4}\,.$$

Here $\gamma(a;r)$ denotes the circle, centre a and radius r, oriented anticlockwise.

(c) (*i*) Find the residues of the function

$$f(z) = \frac{1}{z^2 - 5z + 1}\,.$$

(*ii*) Use contour integration and the result of (*i*) to determine

$$\int_0^{2\pi} \frac{d\theta}{5 - 2\cos\theta}\,.$$

3. (a) Find the Taylor series up to and including the term z^5 (5-jet) of each of the following functions:

$$(i) \quad \sinh(2z^4 + z)\,; \qquad\qquad (ii) \quad \frac{1}{\cos^2(z)}\,.$$

(b) Give the definition of a simple pole singularity.

Determine the type of singularity exhibited by the function

$$f(z) = \frac{\sin 2z}{(2z+\pi)z^3 \cos z}$$

at (i) $z = -\pi/2$, (ii) $z = 0$, (iii) $z = \pi/2$. If the singularity is a simple pole determine the residue, and if it is removable determine the limiting value.

4. (a) Write down the Laplace equation for a function $v(x,y)$ of two real variables. Show that the imaginary part of a holomorphic function satisfies the Laplace equation.

(b) For what value of a real constant $k > 0$ can the function

$$v(x,y) = e^{2x} \cdot \sin(ky)$$

be the imaginary part of a function $f(z) = f(x+iy)$ holomorphic on $\mathbf{C}$?

(c) Write down the expression of the derivative of a holomorphic function in terms of the partial derivatives of its imaginary part only. Apply this to determine the derivative $g(z) = f'(z)$ of a holomorphic function with the imaginary part v you obtained in (b). Express g in terms of z only (not x and y).

(d) Show that, for the function g found in (c),

$$g(z) = -2 \qquad \Longleftrightarrow \qquad z = \frac{i}{2}\pi(2n+1) \ \text{ for some integer } n\,.$$

5. (i) Prove that, if $z^2 \neq -i$, then

$$\sum_{n=0}^{m} \left(iz^2\right)^n = \frac{iz^2(i^m z^{2m}) - 1}{iz^2 - 1}\,.$$

Hence show that $\sum_{n=0}^{\infty} i^n z^{2n}$ is convergent for $|z| < 1$ and find a formula for the sum of this series.

Assuming that the term-by-term differentiation is valid, find the sum of the series

$$\sum_{n=1}^{\infty} n i^n z^{2n}$$

for $|z| < 1$.

(ii) What is the radius of convergence of a power series?

Find the radius of convergence R of the series

$$\sum_{n=0}^{\infty} \frac{(2i)^n}{n^2+1} z^n\,.$$

Determine the convergence or divergence of this series for $|z| = R$. Make sure that your argument applies to all z with $|z| = R$.

6. (a) State Laurent's Theorem.

(b) Sketch the annulus $\{z \in \mathbf{C} : 2 < |z - 2i| < 3\}$, and mark the poles of the function

$$f(z) = \frac{10}{z^2 - 5iz}$$

on your sketch.

(c) Find the Laurent expansion of $f(z)$ valid in the above annulus.

(d) Determine whether this expansion converges at $z = -2 + i$.

7. (a) State the Estimation Theorem and give a proof of the first inequality.

(b) Sketch the path $\gamma_R : [-\pi, 0] \to \mathbf{C}$ defined by $\gamma_R(t) = Re^{-it}$, where $R > 0$.

Prove that

$$\int_{\gamma_R} \frac{e^{iz}}{(z^2+1)^2}\, dz \to 0 \quad \text{as} \quad R \to \infty\,.$$

By integrating $\dfrac{e^{iz}}{(z^2+1)^2}$ along a suitable contour, find $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(x)}{(x^2+1)^2}\, dx\,.$

8. (a) State Liouville's Theorem.

(b) Find the principal value of the integral

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{(x+1)(x^2+2x+2)}\, dx$$

by integrating an appropriate holomorphic function round a large semicircle in the upper half-plane, indented at $z = -1$.

State without proof any results you use on the limiting values of the integrals round the semicircular parts of the contour.