1. (a) (*i*) Let $f(x+iy) = u(x,y) + iv(x,y)$, where x, y, u and v are real. Derive the Cauchy-Riemann equations which hold where f is holomorphic.

(*ii*) Find the real and imaginary parts of the function

$$f(z) = |z|^2 - z^2 \, .$$

Show that f satisfies both Cauchy-Riemann equations if and only if $z = 0$.

(*iii*) Find a holomorphic function on $\mathbf{C}$ with the imaginary part $v(x,y) = 3x^2 - 3y^2$.

(b) Sketch the path $\gamma : [-1,2] \to \mathbf{C}$ given by

$$\gamma(t) = \begin{cases} it\,, & -1 \le t \le 1\,, \\ 2t - 2 + i\,, & 1 \le t \le 2\,. \end{cases}$$

Evaluate $\int_\gamma \operatorname{Im} z \, dz$.

2. (a) State Cauchy's Theorem.

(b) Evaluate the following integrals, in each case giving a brief justification for your answer:

$$\int_{\gamma(6;1)} \frac{dz}{z^2+36}\,; \qquad \int_{\gamma(-5i;2)} \frac{dz}{z^2+36}\,; \qquad \int_{\gamma(0;10)} \frac{dz}{z^2+36}\,.$$

Here $\gamma(a;r)$ denotes the circle, centre a and radius r, oriented anticlockwise.

(c) (*i*) Find the residues of the function

$$f(z) = \frac{1}{z^2+3z+1}\,.$$

(*ii*) Use contour integration and the result of (*i*) to determine

$$\int_0^{2\pi} \frac{d\theta}{2\cos\theta + 3}\,.$$

3. (a) Find the 5-jet at 0 (the Taylor series up to and including the term z^5) of each of the following functions:

$$(i) \quad \sin(z^4 - z)\,; \qquad\qquad (ii) \quad \frac{1}{\cosh^2(2z)}\,.$$

(b) Give the definition of a removable singularity.

Determine the type of singularity exhibited by the function

$$f(z) = \frac{\cos z}{(2z - \pi)^2 \cdot \sin 2z}$$

at (i) $z = 0$, (ii) $z = \pi/2$, (iii) $z = -\pi/2$. If the singularity is a simple pole determine the residue, and if it is removable determine the limiting value.

4. (a) Write down the Laplace equation for a function $u(x, y)$ of two real variables. Show that the real part of a holomorphic function satisfies this equation.

(b) For what value of a real constant $k > 0$ can the function

$$u(x, y) = e^{kx} \cdot \sin(2y)$$

be the real part of a function $f(z) = f(x + iy)$ holomorphic on $\mathbf{C}$?

(c) Write down the expression of the derivative of a holomorphic function in terms of the partial derivatives of its real part. Apply this to determine the derivative $g(z) = f'(z)$ of a holomorphic function with the real part v you obtained in (b). Express g in terms of z (not x and y).

(d) Show that, for the function g found in (c),

$$g(z) = 2 \qquad \Longleftrightarrow \qquad z = i\Big(\frac{\pi}{4} + \pi n\Big) \text{ for some integer } n\,.$$

5. (i) Prove that, if $z^4 \neq \frac{1}{16}$, then

$$\sum_{n=0}^{r} \left(16z^4\right)^n = \frac{1 - 16^{r+1} z^{4r+4}}{1 - 16z^4} .$$

Hence show that $\sum_{n=0}^{\infty} 16^n z^{4n}$ is convergent for $|z| < \frac{1}{2}$ and find a formula for the sum of this series.

Assuming that the term-by-term differentiation is valid, find the sum of the series

$$\sum_{n=1}^{\infty} n 16^n z^{4n}$$

for $|z| < \frac{1}{2}$.

(ii) What is the radius of convergence of a power series?

Find the radius of convergence R of the series

$$\sum_{n=0}^{\infty} \frac{5^n}{n^2 + 3} z^n .$$

Determine the convergence or divergence of this series for $|z| = R$. [Make sure that your argument applies to all z with $|z| = R$.]

6. (a) State Laurent's Theorem.

(b) Sketch the annulus $\{z \in \mathbf{C} : 1 < |z - 4| < 4\}$, and mark the poles of the function

$$f(z) = \frac{10}{z^2 - 5z}$$

on your sketch.

(c) Find the Laurent expansion of $f(z)$ valid in the above annulus.

(d) Determine whether this expansion converges at $z = 1 - 3i$.

7. (a) State the Corollary of the Estimation Theorem and derive it from the Theorem itself.

(b) Sketch the path $\gamma_R : [0, \pi] \to \mathbf{C}$ defined by $\gamma_R(t) = Re^{it}$, where $R > 0$. Prove that

$$\int_{\gamma_R} \frac{e^{4iz}}{(z^2 + 8z + 17)^2}\, dz \to 0 \quad \text{as} \quad R \to \infty\,.$$

By integrating $e^{4iz}/(z^2 + 8z + 17)^2$ along a suitable contour, find

$$\int_{-\infty}^{\infty} \frac{\cos(4x)}{(x^2 + 8x + 17)^2}\, dx\,.$$

8. (a) State Cauchy's Residue Theorem.

(b) Find the principal value of the integral

$$\int_0^{\infty} \frac{x \sin(2x)}{(x^2 - c^2)(x^2 + 16)}\, dx$$

by integrating an appropriate holomorphic function round a large semicircle in the upper half-plane, indented both at $z = c$ and $z = -c$.

State without proof any results you use on the limiting values of the integrals round the semicircular parts of the contour.