1. (a) (*i*) Let $f(x+iy) = u(x,y) + iv(x,y)$, where x, y, u and v are real. Derive the Cauchy-Riemann equations which hold where f is holomorphic.

(*ii*) Find the real and imaginary parts of the function

$$f(z) = (z-2)(z-\overline{z})\,.$$

Show that f satisfies both Cauchy-Riemann equations if and only if $z=2$.

(*iii*) Find a holomorphic function on $\mathbf{C}$ with the imaginary part $v(x,y) = x^3 - 3xy^2$.

(b) Sketch the path $\gamma : [-\frac{1}{4}, \pi] \to \mathbf{C}$ given by

$$\gamma(t) = \begin{cases} 2t+1\,, & -\frac{1}{4} \le t \le 0\,, \\ e^{-it}\,, & 0 \le t \le \pi\,. \end{cases}$$

Evaluate $\int_\gamma \overline{\left(\frac{1}{z}\right)}\,dz\,.$

2. (a) Formulate Cauchy's Theorem.

(b) Evaluate the following integrals, giving brief reasons:

$$\int_{\gamma(-\pi i;2)} \frac{dz}{z^2+4iz}\,; \qquad \int_{\gamma(\pi i;2)} \frac{dz}{z^2+4iz}\,; \qquad \int_{\gamma(-1;5)} \frac{dz}{z^2+4iz}\,.$$

Here $\gamma(a;r)$ denotes the circle, centre a and radius r, oriented anticlockwise.

(c) (*i*) Find the residues of the function

$$f(z) = \frac{1}{20z^2 - 41z + 20}\,.$$

(*ii*) Use contour integration and the result of (*i*) to determine

$$\int_0^{2\pi} \frac{dt}{40\cos t - 41}\,.$$

3. (a) Find the 5-jet at 0 (the Taylor series up to and including the term z^5) of each of the following functions:

$$(i) \quad e^{1-\cos z^2}\,; \qquad\qquad (ii) \quad \frac{1}{2+\sinh^2(2z)}\,.$$

(b) Formulate the definition of a removable singularity.

Determine the type of singularity exhibited by the function

$$f(z) = \frac{\sin z \cdot \cot 2z}{(z+\frac{\pi}{2})^3}$$

at (i) $z = 0$, (ii) $z = -\pi/2$, (iii) $z = \pi/2$. If the singularity is a simple pole determine the residue, and if it is removable determine the limiting value.

4. (a) Write down the Laplace equation for a function $u(x, y)$ of two real variables. Show that the real part of a holomorphic function satisfies this equation.

(b) For what value of the positive constant k can the function

$$u(x, y) = \cosh x \cdot \cos(ky)$$

be the real part of a function $f(z) = f(x + iy)$ holomorphic on $\mathbf{C}$?

(c) Write down the expression of the derivative of a holomorphic function in terms of the partial derivatives of its real part. Apply this to determine the derivative $g(z) = f'(z)$ of a holomorphic function with the real part u you obtained in (b). Express g in terms of z (not x and y).

(d) Show that, for the function g found in (c),

$$g(z) = 0 \qquad \Longleftrightarrow \qquad z = \pi n i \ \text{ for some integer } n\,.$$

5. (i) Prove that, if $z \neq 0$ and $z^3 \neq 2$, then

$$\sum_{n=0}^{r} \left(\frac{2}{z^3}\right)^n = \frac{1 - \frac{2^{r+1}}{z^{3r+3}}}{1 - \frac{2}{z^3}}.$$

Hence show that $\sum_{n=0}^{\infty} 2^n z^{-3n}$ is convergent for $|z| > 2^{1/3}$ and find a formula for the sum of this series.

Assuming that the term-by-term differentiation is valid, find the sum of the series

$$\sum_{n=1}^{\infty} n 2^n z^{-3n}$$

for $|z| > 2^{1/3}$.

(ii) What is the radius of convergence of a power series?

Find the radius of convergence R of the series

$$\sum_{n=0}^{\infty} \frac{n 4^n}{\sqrt{n^2+1}} z^n .$$

Determine the convergence or divergence of this series for $|z| = R$. [Make sure that your argument applies to all z with $|z| = R$.]

6. (a) Formulate Laurent's Theorem.

(b) Sketch the annulus $\{z \in \mathbf{C} : 3 < |z+4| < 5\}$, and mark the poles of the function

$$f(z) = \frac{8}{z^2 + 6z - 7}$$

on your sketch.

(c) Find the Laurent expansion of $f(z)$ valid in the above annulus.

(d) Determine whether this expansion converges at $z = -2 + 3i$.

7. (a) Derive the Corollary of the Estimation Theorem from the Theorem itself.

(b) Sketch the path $\gamma_R : [0, \pi] \to \mathbf{C}$ defined by $\gamma_R(t) = Re^{it}$, where $R > 0$. Prove that

$$\int_{\gamma_R} \frac{e^{4iz}}{(z^2 - 2z + 2)^2}\, dz \to 0 \quad \text{as} \quad R \to \infty\,.$$

By integrating $e^{4iz}/(z^2 - 2z + 2)^2$ along a suitable contour, find

$$\int_0^\infty \frac{\sin(4x)}{(x^2 - 2x + 2)^2}\, dx\,.$$

8. (a) Formulate Cauchy's Residue Theorem.

(b) Let c be a positive constant. Find the principal value of the integral

$$\int_0^\infty \frac{\cos x}{x^4 - c^4}\, dx$$

by integrating an appropriate holomorphic function round a large semicircle in the upper half-plane, indented both at c and $-c$.

State without proof any results you use on the limiting values of the integrals round the semicircular parts of the contour.