1.

a) Let (x_n) be the sequence defined by $x_0 = 2$ and

$$x_{n+1} = \frac{x_n}{2} + \frac{3}{2x_n}$$

for each $n \geq 0$. Prove that (x_n) is a decreasing sequence which tends to $\sqrt{3}$ as $n \to \infty$. (You may use any results from the lectures without proof, provided that you state them clearly.) [9 marks]

b) Show that $|x_n - \sqrt{3}| < (1/8)^n$. How large should n be to ensure that x_n agrees with $\sqrt{3}$ to 1000 decimal places? [6 marks]

c) State the completeness axiom of the real numbers, and use it to prove that any increasing real-valued sequence (x_n) which is bounded above converges. [5 marks]

2.

(In this question, you may find it helpful to use the formulae: $p_0 = a_0$, $p_1 = a_1 a_0 + 1$, $p_n = a_n p_{n-1} + p_{n-2}$ for $n \geq 2$; $q_0 = 1$, $q_1 = a_1$, $q_n = a_n q_{n-1} + q_{n-2}$ for $n \geq 2$.)

a) Calculate the value of, and the first four convergents to, the continued fraction $[3, 1, 3, 1, 3, 1, \ldots]$. [5 marks]

b) Let the continued fraction expansion of $\pi = 3.14159265\ldots$ be given by $[a_0, a_1, a_2, a_3, \ldots]$. Using your calculator, determine a_n for $0 \leq n \leq 3$ (you do not need to write anything down other than the values of each a_n). Hence calculate the first 4 convergents to π. In what sense are the convergents the "best possible" rational approximations of π? [7 marks]

c) Let (a_n) be any sequence of positive integers, and let

$$\frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n]$$

for each $n \geq 0$. Given that

$$\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{(-1)^{n-1}}{q_{n-1}q_n} \qquad \text{and} \qquad \frac{p_n}{q_n} - \frac{p_{n-2}}{q_{n-2}} = \frac{(-1)^n a_n}{q_{n-2}q_n}$$

for all integers $n \geq 2$, show that the sequence (p_n/q_n) converges. [8 marks]

3.

a) What does it mean for an infinite set S to be countable? [3 marks]

b) Show that $\mathbf{Q}$ is countable. [5 marks]

c) Show that if $S_1, S_2, \ldots, S_n$ are countable infinite sets, then so is their union

$$\bigcup_{i=1}^{n} S_i.$$

[6 marks]

d) Show that the set S of all subsets of the natural numbers is uncountable. [6 marks]

4.

a) State Sharkovsky's theorem (concerning the possible sets of periods of continuous maps $f\colon [0,1] \to [0,1]$). [4 marks]

b) Determine the Markov graph of the period 7 pattern $(1\,4\,6\,2\,5\,3\,7)$. Suppose that $f\colon [0,1] \to [0,1]$ is a continuous map with a periodic orbit of this pattern: what other periods of orbits must f have? [9 marks]

c) Consider the map $f\colon [0,1] \to [0,1]$ given by $f(x) = 4x(1-x)$; and the division of $[0,1]$ given by $J_1 = [0, 1/2]$ and $J_2 = [1/2, 1]$. Determine the itinerary of the point $x = 1/4$. Sketch the graph of $f(x)$, and indicate the following sets on the x-axis:

i) The set A_0 of points x whose itinerary $k(x)$ starts with 2.

ii) The set A_1 of points x whose itinerary $k(x)$ starts with 21.

iii) The set A_2 of points x whose itinerary $k(x)$ starts with 212.

[7 marks]

5. For each $r \in \mathbf{R}$, let $f_r : \mathbf{R} \to \mathbf{R}$ be the map defined by

$$f_r(x) = r - x^2.$$

a) Determine the fixed and period 2 points of f_r. [8 marks]

b) Determine the range of values of r for which each fixed and period 2 point of f_r is stable. (You are not required to analyse the cases in which the multiplier is equal to 1 or -1.) [12 marks]

6.

a) What does it mean for a square matrix P to be a stochastic matrix? Under what conditions is it true that there is a unique vector $\mathbf{x}$ with the property that $P^n\mathbf{x}_0 \to \mathbf{x}$ as $n \to \infty$ for all probability vectors $\mathbf{x}_0$? Give an example of a stochastic matrix P for which there is no such unique limit vector $\mathbf{x}$, and prove that the matrix P you write down has this property.

[6 marks]

b) Determine the unique limit vector $\mathbf{x}$ for the matrix

$$P = \begin{pmatrix} \frac{1}{3} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{2} \end{pmatrix}.$$

[7 marks]

c) State the contraction mapping theorem (concerning fixed points of functions $f\colon A \to A$, where A is a closed subset of $\mathbf{R}^n$). Give an example of a contraction map $f\colon (0,1) \to (0,1)$ which has no fixed points. Show that the map $f\colon [1,\infty) \to [1,\infty)$ defined by

$$f(x) = x + \frac{1}{x}$$

satisfies $|f(x) - f(y)| < |x - y|$ for all $x \neq y$, but has no fixed points. Why does this not contradict the contraction mapping theorem? [7 marks]

7.

a) What does it mean for a function $f\colon \mathbf{R} \to \mathbf{R}$ to be odd? Explain why an odd periodic function has no constant or cosine terms in its Fourier series expansion. [3 marks]

b) Calculate the Fourier series expansion of the 2π-periodic function $f(t)$ defined for $t \in [-\pi, \pi)$ by

$$f(t) = \begin{cases} -1 & \text{if } -\pi \leq t < 0 \\ 1 & \text{if } 0 \leq t < \pi. \end{cases}$$

[7 marks]

c) By applying the Fourier series theorem at $t = \pi/2$, show that

$$\sum_{r=0}^{\infty} \frac{(-1)^r}{2r+1} = \frac{\pi}{4}.$$

[5 marks]

d) Define what it means for a series

$$\sum_{r=1}^{\infty} F_r(t)$$

of functions $F_r\colon \mathbf{R} \to \mathbf{R}$ to converge *pointwise*, and to converge *uniformly*, to a function $F(t)$. What function does the Fourier series expansion in part b) converge to pointwise? Is the convergence uniform? [5 marks]

8. The Fourier series expansion of the function t ($t \in [-\pi, \pi)$) is

$$\sum_{r=1}^{\infty} \frac{2(-1)^{r+1}}{r} \sin rt.$$

a) By integrating the Fourier series expansion of t term by term, obtain the Fourier series expansion of t^2 ($t \in [-\pi, \pi)$). [5 marks]

b) By integrating the Fourier series expansion you obtained in part a) term by term, obtain the Fourier series expansion of t^3 ($t \in [-\pi, \pi)$). [7 marks]

c) By applying Parseval's theorem to the Fourier series expansion of t^2, evaluate

$$\sum_{r=1}^{\infty} \frac{1}{r^4}.$$

[8 marks]