PAPER CODE NO.
MATH728

EXAMINER:

DEPARTMENT: TEL. NO:

THE UNIVERSITY *of* LIVERPOOL

SUMMER 2006 EXAMINATIONS

Bachelor of Science: Year 3
Master of Mathematics: Year 3

MECHANICS AND RELATIVITY

TIME ALLOWED : Two Hours and a Half

INSTRUCTIONS TO CANDIDATES

You may attempt all questions. Your best answers to FIVE questions only will be taken into account. The marks shown against the sections indicate their relative weights.

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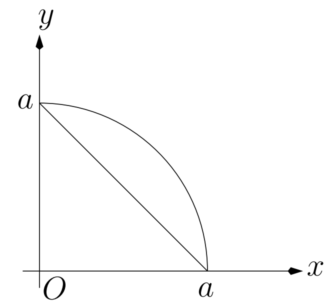
1. **i**. A rigid uniform lamina, bounded below by a straight line and above by a quarter of a circle, lies in the $xy-$plane, as shown in the figure.



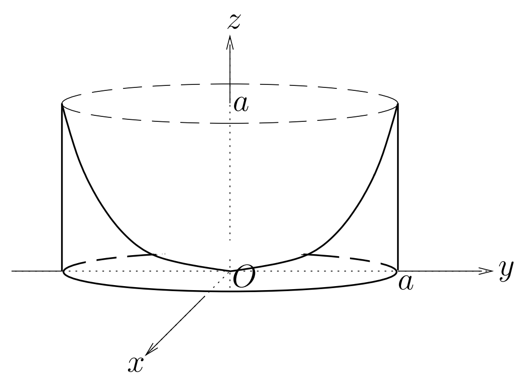
Find the coordinates of the centre of mass of the lamina.

[5 marks]

The lamina is now rotated through 2π about $x = 2a$. Show, by the use of the Theorem of Pappus-Guldin or otherwise, that the volume swept out is given by $\dfrac{(3\pi - 7)\pi}{3}\, a^3$. [2 marks]

<u>`Hint`</u> `Centre of mass:` $\mathbf{r}_G = \dfrac{1}{A}\displaystyle\int_A \mathbf{r}\, dA$

ii. A rigid uniform solid of revolution of mass M is placed symmetrically about the $z-$axis. As shown in the figure, the solid is enclosed by a cylinder and bounded above by a paraboloid and below by the $xy-$plane. In terms of cylindrical polar coordinates $x = r\cos(\theta)$, $y = r\sin(\theta)$, the equation of the curved surface of the cylinder is given by $r = a$, whereas the paraboloid's curved surface is given by $z = r^2/a$, where $0 \le z \le a$, $0 \le \theta \le 2\pi$.



Identify the principal axes of inertia for this solid at the origin O, stating <u>briefly</u> your reasons. [2 marks]

Show that the volume of the solid is $\pi a^3/2$. [3 marks]

Hence, deduce that the moment of inertia of the solid about the $z-$axis, i.e. I_{Oz}, is $2Ma^2/3$. [3 marks]

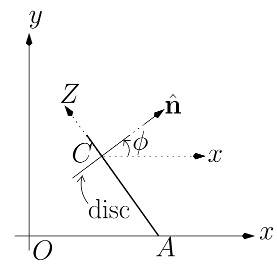
Further, deduce that the moment of inertia of the solid about the $y-$axis is $Ma^2/2$.

[5 marks]

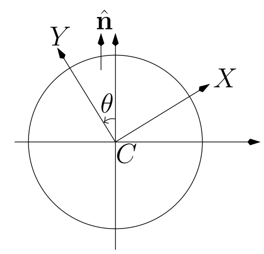
<u>`Hint`</u> `Inertia matrix:` $\dfrac{M}{V}\displaystyle\int_V \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix} dV.$

2. **i.** Define the angular velocity of a rigid uniform body in terms of an arbitrary body vector. [2 marks]

ii. A rigid uniform circular disc is attached to a rigid uniform thin rod at its centre C, about which it rotates, as shown in the top figure. The rod swings freely, about its end A, a fixed point along the $x-$axis, remaining in the $xy-$plane at all times. The choice of the fixed coordinate axes $Oxyz$, forming a right-handed frame, is also shown in the top figure.



$\hat{\mathbf{n}}$ is a unit vector in the $xy-$plane and ϕ is the angle between the $x-$axis and the plane of the disc, at time t. CX, CY, CZ are mutually perpendicular body axes, again forming a right-handed frame. A top-on view of the disc is shown in the bottom figure, showing CX, CY lying on the disc. As shown, θ is the angle between $\hat{\mathbf{n}}$ and CY at time t.



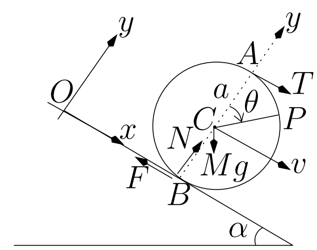
Identify the unit vector along the horizontal axis in the bottom figure. [1 mark]

Hence, show that, at time t the angular velocity of the disc is given by

$$\boldsymbol{\omega} = \dot{\phi}\cos(\theta)\,\mathbf{I} - \dot{\phi}\sin(\theta)\,\mathbf{J} + \dot{\theta}\,\mathbf{K},$$

where $\mathbf{I}, \mathbf{J}$ are unit vectors along CX, CY and $\mathbf{K}$ is a unit vector along the axis of the rod CZ. [6 marks]

iii. A rigid uniform disc of radius a and mass M is rolling down <u>without slipping</u> on a plane which is inclined at an angle α to the horizontal, as shown in the figure. B is the instantaneous point of contact between the disc and the inclined plane, and C is the centre of the disc, which moves with constant speed v in the positive $x-$direction.



The choice of the coordinate axes, forming a right handed frame, is also shown in the figure. The motion of the disc is restricted to the $xy-$plane at all times. θ is the angle between Cy and $\mathbf{CP}$, were P is a fixed point on the rim of the disc.

In addition to frictional, normal and gravitational forces, a thrust T, parallel to the positive $x-$axis, is applied at point A, as shown.

Show that the angular momentum of the disc about the point B is

$\mathbf{L}_B = -(3Ma^2\dot{\theta}/2)\,\mathbf{k}.$ [4 marks]

Hence, using the equation for rotational motion about B and the no-slip condition, deduce that

$\dfrac{2}{3}\dfrac{Mg\,\sin(\alpha)+2T}{Ma} = \ddot{\theta}.$ [3 marks]

Finally, show that the displacement of the disc is given by

$x(t) = \dfrac{1}{3}\dfrac{Mg\,\sin(\alpha)+2T}{Ma}\,t^2,$

given that the disc starts rolling down from rest from the origin O.

[4 marks]

<u>`Hint`</u>

`Theorem of Parallel Axes:` $I_{\parallel} = I_G + Md^2$

`Equations of motion:` $\sum_i \mathbf{CP}_i \times \mathbf{F}_i = M(\mathbf{CG} \times \dot{\mathbf{v}}_C) + \dot{\mathbf{L}}_C$

`Angular velocity:` $\mathbf{v}_P = \mathbf{v}_C + \boldsymbol{\omega} \times \mathbf{CP}$

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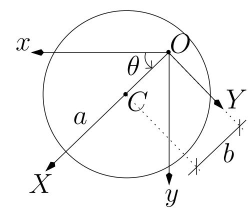
3. **i.** The figure shows a rigid uniform disc of mass M and radius a, smoothly hinged at a point O, a distance b away from the centre of the disc C. The disc rotates about this point, remaining in the $xy-$plane at all times. Ox is the horizontal axis, Oy is vertically downwards and $Oxyz$ forms a fixed right-handed frame. $OXYZ$, the mutually perpendicular body axes, also form a right-handed frame. At time t, θ is the angle between OX and Ox.

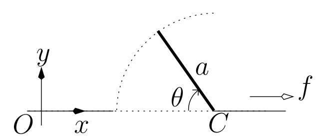


The disc is released at time $t = 0$, from $\theta = 0$ with $\dot{\theta} = \Omega$. Show, by applying the law of conservation of energy, that $\dot{\theta} = [\Omega^2 + \dfrac{2Mgb}{I_{Oz}} \sin(\theta)]^{1/2}$, where I_{Oz} is the moment of inertia of the disc at O about the $z-$axis.

[9 marks]

<u>`Hint`</u> `Kinetic energy:` $T_0 = \dfrac{1}{2} M|\mathbf{v}_0|^2 + \dfrac{1}{2} \boldsymbol{\omega} \cdot \mathbf{L}_0$

ii. The figure shows the vertical cross-section of the top of a railway carriage which has a horizontal roof with a square trapdoor of mass M, side length a and uniform thickness, hinged smoothly to the roof at C and swings freely in the $xy-$plane. The choice of the coordinate axes is also shown in the figure.



Assuming that the carriage is moving with constant acceleration f in the positive $x-$direction, use the equation of motion involving the angular momentum of the trapdoor about the point C to show that:

$$I\ddot{\theta} = -\frac{Maf}{2} \sin(\theta) - \frac{Mga}{2} \cos(\theta),$$

where θ is the angle between the trapdoor and its closed position at time t, and I is its moment of inertia about the horizontal axis through C.

[5 marks]

Now, integrate the above equation of motion with respect to time t and hence show that when the trapdoor closes, the magnitude of its angular velocity will be $[Ma(f + g)/I]^{1/2}$.

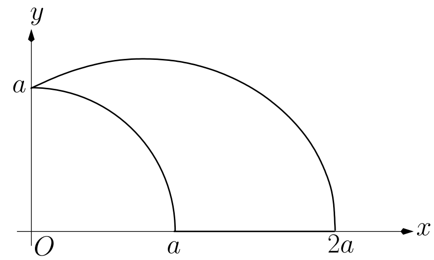
[6 marks]

<u>`Hint`</u>

`Equations of motion:` $\sum_i \mathbf{CP}_i \times \mathbf{F}_i = M(\mathbf{CG} \times \dot{\mathbf{v}}_C) + \dot{\mathbf{L}}_C$

4. **i.** A rigid uniform lamina lies in the first quadrant of the $xy-$plane, bounded below by the circle $r = a$ and above by the cardioid $r = a + a\cos(\theta)$, as shown in the figure.

Find the moment of inertia I_{Oy}. [7 marks]

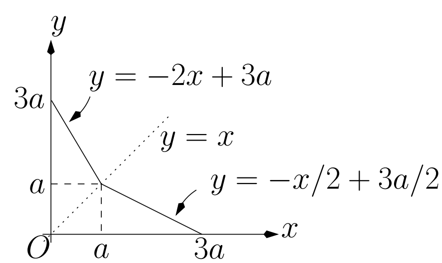


ii. The figure shows a rigid uniform lamina of mass M, bounded by the lines $y = -2x + 3a$, $y = -x/2 + 3a/2$, the $x-$axis and the $y-$axis. The lamina is clearly symmetric about the line $y = x$. Ox, Oy are the body axes and the axis Oz is perpendicular to the $xy-$plane, which form a right handed frame.
Obtain the Cartesian equations of the principal axes of inertia at the origin O for the lamina by considering its symmetry properties.

[2 marks]

Find the moments of inertia I_{Ox} and I_{Oy} <u>only</u>.

[5 marks]



It can be shown that the inertia matrix for the lamina, evaluated at C relative to the body axes, is given by:

$$\frac{Ma^2}{12}\begin{pmatrix} 14 & -5 & 0 \\ -5 & 14 & 0 \\ 0 & 0 & 28 \end{pmatrix}.$$

Write down column matrices representing vectors parallel to the principal axes you found. Hence, obtain the principal moments of inertia corresponding to each principal axes. [6 marks]

<u>Hint</u> `You may refer to the inertia matrix given in question 1.`

`You may also use the formula` $I_n = \displaystyle\int_0^{\pi/2} \cos^n(\theta)\, d\theta = \frac{n-1}{n} I_{n-2}$.

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5. At time t, the components of the angular velocity of a uniform rigid body along its principal axes of inertia GX, GY, GZ at its centre of mass G, are $\omega_1(t), \omega_2(t)$ and $\omega_3(t)$ respectively. The corresponding principal moments of inertia at G are I_1, I_2 and I_3.

   Given that the angular momentum of the body at C can be represented by $\mathbf{L}_C = I_1\omega_1\mathbf{I} + I_2\omega_2\mathbf{J} + I_3\omega_3\mathbf{K}$, where $\mathbf{I}, \mathbf{J}, \mathbf{K}$ are the unit vectors along the body axes, find the derivative of $\mathbf{L}_C$ with respect to time t, to derive the Euler form.

   [5 marks]

   If the principal moments of inertia at G are chosen to be $I, 2I$ and αI respectively, where α is a constant parameter and all the external forces on the body act at G, show that the angular momentum of the body at G is constant. [2 marks]

   Hence use the Euler's form to verify that the Euler equations for the three-dimensional motion of the body can be written as

$$\dot{\omega}_1 = (2-\alpha)\,\omega_2\omega_3, \quad 2\,\dot{\omega}_2 = (\alpha-1)\,\omega_3\omega_1, \quad \alpha\,\dot{\omega}_3 = -\omega_1\omega_2.$$

   [2 marks]

   Given that $\alpha = 1$, show, by using the Euler's equations of motion, that

$$\omega_1\dot{\omega}_1 + \omega_3\dot{\omega}_3 = 0.$$

   [2 marks]

   By integrating this equation with respect to time show that the magnitude of the angular velocity is a constant of the motion. [3 marks]

   Now, assuming that α can take all values $\alpha > 0$, let $\omega_1 = \varepsilon_1(t), \omega_2 = s + \varepsilon_2(t)$ and $\omega_3 = \varepsilon_3(t)$, where s is constant and ε_i, $i = 1, 2, 3$ are small perturbations from equilibrium at time t. Deduce that the equilibrium state is stable for $\alpha < 2$ and unstable for $\alpha > 2$.

   [6 marks]

   <u>`Hint`</u>

   `Equations of motion:` $\sum_i \mathbf{CP}_i \times \mathbf{F}_i = M(\mathbf{CG} \times \dot{\mathbf{v}}_C) + \dot{\mathbf{L}}_C$

   `Euler form:`

$$\dot{\mathbf{L}}_C = [I_1\,\dot{\omega}_1 - (I_2 - I_3)\,\omega_2\omega_3]\,\mathbf{I} + [I_2\,\dot{\omega}_2 - (I_3 - I_1)\,\omega_3\omega_1]\,\mathbf{J} + [I_3\,\dot{\omega}_3 - (I_1 - I_2)\,\omega_1\omega_2]\,\mathbf{K}$$

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6. A symmetric spinning top moves under gravity about a stationary fixed pivot O on its axis of symmetry. The centre of mass G of the top lies on this symmetry axis at a distance a from O.

   Draw a clearly labelled figure to define the conventional Euler angles θ, ϕ and ψ which specify the position of the top relative to fixed space axes $Oxyz$, where Oz is vertically upwards. What do the time derivatives of Euler angles characterise? [3 marks]

   It may be assumed that the mass and the principal moments of inertia of the top are such that its motion at time t is given by the equations:

$$\begin{aligned} \dot{\psi} + \dot{\phi}\cos(\theta) &= s, & (i)\\ s\cos(\theta) + \lambda\,\dot{\phi}\sin^2(\theta) &= A, & (ii)\\ \dot{\theta}^2 + \dot{\phi}^2\sin^2(\theta) + \mu\,a\cos(\theta) &= B, & (iii) \end{aligned}$$

   where $s \neq 0, A$ and B are constants of motion, and $\lambda = I_1/I_3$, $\mu = 2Mg/I_1$ are constants. Here, I_1, I_3 denote principal moments of inertia, M mass and g gravitational acceleration.

   What can you say about the sign of the constants λ and μ?

   [1 mark]

   The above equations of motion allow existence of a state of dynamical equilibrium in which the top precesses indefinitely, with OG rotating about Oz with constant angular velocity Ω, and is inclined at a constant angle α to Oz, where $0 < \alpha < \pi$. Verify this, explaining briefly.

   [2 marks]

   Now, by differentiating equations (ii) and (iii) with respect to time, eliminate the terms involving $\ddot{\phi}$, and use equation (i) to show that

$$2\lambda\ddot{\theta} = [2(\lambda - 1)\cos(\theta)\,\dot{\phi}^2 - 2\dot{\psi}\dot{\phi} + \lambda\mu a]\sin(\theta).$$

   [7 marks]

   What does $\lambda = 1$ correspond to, and is it sensible to assume this? Justify your answer briefly.

   [2 marks]

   What is the condition on λ for the top to precess steadily?

   [2 marks]

   Find two approximate speeds of precession.

   [3 marks]

7. **a.** Within the theory of special relativity, the $y-$ and $z-$axes of an inertial frame S are parallel to the corresponding axes of another inertial frame S'. The $x-$axes of S and S' are collinear and S' moves with constant velocity v relative to S.

Two events which take place at different spatial locations in S are observed from S' to occur at the same time t' relative to S'. Show that the events cannot occur simultaneously with respect to S.

[4 marks]

b. A spaceship A leaves Earth, moving on a straight line with constant velocity $u << c$, where c is the speed of light in vacuo. When A is observed from Earth to have travelled a distance d, spaceship B leaves Earth, moving in the same straight line as spaceship A and with velocity $5u$.

i. Use Lorentz transformation to show that the velocity of B relative to A is approximately given by $4u\,(1 + 5u^2/c^2)$.

[5 marks]

ii. Assume that Event 1 occurs when B leaves Earth, with A a distance d from Earth, and that Event 2 occurs when B catches up with A. Show that the time interval between these events, as observed from Earth, is $d/4u$, whereas this time interval, as observed from A is given by

$$\frac{d}{4u}\sqrt{1 - u^2/c^2}.$$

[7 marks]

iii. Use the above results to show that the distance travelled by B from Earth before catching up with A is observed from A to be $d\,(1 + 4u^2/c^2)$.

[4 marks]

<u>`Hint`</u>

`Lorentz transformation:`

$\Delta x' = \gamma(\Delta x - v\Delta t)$, $\Delta t' = \gamma(\Delta t - v\Delta x/c^2)$ `where` $\gamma = 1/\sqrt{1 - v^2/c^2}$.