MATH727 MATHEMATICAL METHODS FOR NON-PHYSICAL SYSTEMS

JANUARY 2006

1.

$$U(0,N) = 5N + 2 > U(N,0) = 3N + 2$$

so N cups of coffee preferred to N cups of tea.

Indifference curves given by

$$\begin{aligned} y =& \frac{U_0 - 3x - 2}{x+5} = \frac{U_0 - 3(x+5) + 13}{x+5} \\ =& \frac{U_0 + 13}{x+5} - 3. \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} =& -\frac{U_0 + 13}{(x+5)^2} < 0, \\ \frac{d^2y}{dx^2} =& 2\frac{U_0 + 13}{(x+5)^3} > 0, \end{aligned}$$

$\Rightarrow$ indifference curves are sloping downwards and convex.

Budget constraint touches indifference curve where

$$\begin{aligned} \frac{\frac{\partial U}{\partial x}}{\frac{\partial U}{\partial y}} =& \frac{p}{q} \\ \frac{y+3}{x+5} = \frac{p}{q} \Rightarrow& y = \frac{p}{q}(x+5) - 3. \end{aligned}$$

Substituting into budget constraint $px + qy = 12$ we have

$$\begin{aligned} 2px + 5p - 3q = 12 \Rightarrow x =& \frac{12 - 5p + 3q}{2p} = \frac{12 + 3q}{2p} - \frac{5}{2} \\ \Rightarrow y =& \frac{12 + 5p - 3q}{2q} = \frac{12 + 5p}{2q} - \frac{3}{2}. \end{aligned}$$

$$\epsilon_x = p\frac{2p}{12 - 5p + 3q}\left(-\frac{12 + 3q}{2p^2}\right) = \frac{12 + 3q}{5p - 12 - 3q}$$

$$\Rightarrow \epsilon_x + 1 = \frac{5p}{5p - 12 - 3q} < 0 \quad \text{if} \quad 5p - 3q < 12.$$

Similarly

$$\epsilon_y = \frac{12+5p}{3q-12-5p}.$$

2. (a) $c(x,y)$ is minimised where

$$\frac{\frac{\partial q}{\partial x}}{\frac{\partial q}{\partial y}} = \frac{\frac{\partial c}{\partial x}}{\frac{\partial c}{\partial y}}$$

$$\frac{\frac{1}{3}(x+1)^{-\frac{2}{3}}(y+2)^{\frac{2}{3}}}{\frac{2}{3}(x+1)^{\frac{1}{3}}(y+2)^{-\frac{1}{3}}} = \frac{27}{2}$$

$$\frac{1}{2}\frac{y+2}{(x+1)} = \frac{27}{2} \Rightarrow y+2 = 27(x+1)$$

$$\Rightarrow 27^{\frac{2}{3}}(x+1) = 12 \Rightarrow 9(x+1) = 12 \Rightarrow x = \tfrac{1}{3}, \quad y = 34$$

$$\Rightarrow c = 27x + 2y + 4 = 9 + 68 + 4 = 81.$$

Minimum cost for production level of 12 units is 81 units.

(b)

$$C(q) = q^3 - 6q^2 + 14q + 10.$$

(i) Fixed cost $C(0) = 12$. (ii) $MC(q) = C'(q) = 3q^2 - 16q + 24$.

(iii) $AVC(q) = \frac{C(q)-C(0)}{q} = q^2 - 8q + 24$.

Cease production when $p = \min(AVC)$.

$$AVC'(q) = 2q - 8 = 0 \quad \text{when} \quad q = 4 \Rightarrow p = AVC(4) = 8.$$

(c)

$$C(q) = q^3 - 6q^2 + 14q + 3, \quad D(p) = 22 - p = q \Rightarrow p = 22 - q.$$

Profit given by

$$P(q) = pq - C(q) = (22-q)q - (q^3 - 6q^2 + 14q + 3) = -\,q^3 + 5q^2 + 8q - 3$$

$$\Rightarrow P'(q) = -3q^2 + 10q + 8 = -(3q+2)(q-4) = 0 \quad \text{for} \quad q = -\tfrac{2}{3}, \quad 4.$$

Take the +ve solution $q = 4$. Then $p = 22 - q = 18$.

$$P''(q) = -6q + 10 < 0 \quad \text{for} \quad q = 4.$$

So we have a local maximum. Also

$$P(4) = -4^3 + 5.4^2 + 8.4 - 3 = 45 > P(0) = -3.$$

So $q = 4$ is a global maximum.

3.

$$C(q) = q^3 - 4q^2 + 6q + 64 \Rightarrow AVC(q) = q^2 - 4q + 6.$$
$$AVC'(q) = 2q - 4 = 0 \quad \text{when} \quad q = 2 \Rightarrow \min(AVC) = 2.$$

So cease production when $p = \min(AVC) = 2$. For $p \geq 2$,

$$p = C'(q) = 3q^2 - 8q + 6 \Rightarrow 3q^2 - 8q + 6 - p = 0$$
$$\Rightarrow q = \frac{8 \pm \sqrt{64 - 12(6-p)}}{6} = \frac{4 \pm \sqrt{3p-2}}{3}.$$

Take +ve sign for maximum profit. So

$$S(p) = \begin{cases} \frac{4+\sqrt{3p-2}}{3} & \text{if } p \geq 2 \\ 0 & \text{if } p < 2. \end{cases}$$

Equilibrium is when $NS(p) = D(p)$ (N firms) so

$$\frac{4 + \sqrt{3p-2}}{3} = 3 - \tfrac{1}{2}p.$$

$p = 2$ is a solution by inspection, and since $D(p)$ is decreasing and $S(p)$ is increasing, it is unique.

$$p = 2 \Rightarrow q = \frac{1}{N}D(p) = 2 \Rightarrow P(q) = pq - C(q)$$
$$= 4 - (8 - 16 + 12 + 64) = -64.$$

So each firm makes a loss of 64 units.

Production not viable in the long-run for $p < \min(ATC)$.

$$ATC = q^2 - 4q + 6 + \frac{64}{q} \Rightarrow ATC'(q) = 2q - 4 - \frac{64}{q^2}$$
$$= 0 \quad \text{when} \quad q = 4,$$

by inspection. It is a minimum, since

$$ATC''(q) = 2 + \frac{128}{q^3} > 0.$$
$$\min(ATC) = 16 - 16 + 6 + 16 = 22.$$

So minimum price in the long-run is 22 units.

In monopoly case $Nq = N(3 - \frac{1}{2}p) \Rightarrow p = 6 - 2q$. So

$$P(q) = N[pq - C(q)] = N[(6-2q)q - (q^3 - 4q^2 + 6q + 64)] = N[-q^3 + 2q^2 - 64].$$
$$\Rightarrow P'(q) = N[-3q^2 + 4q] = 0 \quad \text{when} \quad q = 0, \quad \tfrac{4}{3}.$$

Take $q = \frac{4}{3} \Rightarrow p = 6 - \frac{8}{3} = \frac{10}{3}$.

4.

$$\begin{aligned}C_1(q_1) =&5 + 4q_1 + q_1^2,\\ C_2(q_2) =&7 + 9q_2 + \tfrac{1}{2}q_2^2,\end{aligned}$$

Profits:

$$\begin{aligned}P_1(q_1, q_2) =&pq_1 - (5 + 4q_1 + q_1^2) = [18 - (q_1 + q_2)]q_1 - (5 + 4q_1 + q_1^2)\\ =& - 2q_1^2 - q_1q_2 + 14q_1 - 5,\\ P_2(q_1, q_2) =&pq_2 - (7 + 9q_2 + \tfrac{1}{2}q_2^2) = [18 - (q_1 + q_2)]q_2 - (7 + 9q_2 + \tfrac{1}{2}q_2^2)\\ =& - q_1q_2 - \tfrac{3}{2}q_2^2 + 9q_2 - 7.\end{aligned}$$

Cournot duopoly $\Rightarrow$ maximise P_1, P_2 wrto q_1, q_2 respectively. So

$$\begin{aligned}\frac{\partial P_1}{\partial q_1} =& - 4q_1 - q_2 + 14 = 0,\\ \frac{\partial P_2}{\partial q_2} =& - q_1 - 3q_2 + 9 = 0,\end{aligned}$$

Then $q_1 = 3$, $q_2 = 2$. So $p = 18 - 3 - 2 = 13$ and $P_1(2, 1) = 13$, $P_2(2, 1) = -1$.

If co-operate, maximise

$$\begin{aligned}P(q_1, q_2) =&P_1(q_1, q_2) + P_2(q_1, q_2)\\ =& - \frac{3}{2}q_1^2 - 2q_1q_2 + 14q_1 - 2q_2^2 + 9q_2 - 12\\ \frac{\partial P_1}{\partial q_1} =& - 4q_1 - 2q_2 + 14 = 0,\\ \frac{\partial P_2}{\partial q_2} =& - 2q_1 - 3q_2 + 9 = 0,\end{aligned}$$

giving $q_1 = 3$, $q_2 = 1$. Then $P_1(2, \frac{1}{2}) = 16$, $P_2(2, \frac{1}{2}) = -\frac{5}{2}$.

If $q_2 = 4$, have

$$\begin{aligned}P_1(q_1, 2) = -2q_1^2 - 4q_1 + 14q_1 - 5 =& - 2q_1^2 + 10q_1 - 6,\\ \frac{\partial P_1}{\partial q_1} =& - 4q_1 + 10 = 0 \quad \text{when} \quad q_1 = \tfrac{5}{2}.\end{aligned}$$

So the first firm lowers its production to $q_1 = \frac{5}{2}$ units.

Then

$$P_2(\tfrac{5}{2}, 4) = -q_1q_2 - \tfrac{3}{2}q_2^2 + 9q_2 - 7 = -\tfrac{5}{2}.4 - \tfrac{3}{2}.4^2 + 6.4 - 4 = -16$$

so the second firm doesn't benefit.

5. (a)

$$\mathbf{x} = \mathbf{x}^e + c_1\mathbf{x}_1 e^{\lambda_1 t} + c_2\mathbf{x}_2 e^{\lambda_2 t},$$

where $\lambda_{1,2}$ are the e-values, $\mathbf{x}_{1,2}$ are the e-vectors. If λ_1 and λ_2 have the same signs, then we have an improper node; stable if the e-values are $-$ve, unstable if $+$ve.

(b)

$$\frac{dn}{dt} = 18n^2 - 3n^3 = -3n^2(n-6) = f(n).$$

The equilibrium densities are $n = 0$ and $n = 6$. The graph of $f(n)$ looks like this:

so $n = 0$ is unstable, $n = 6$ unstable.

Now

$$\frac{dn}{dt} = 18n^2 - 3n^3 - cn = -n(3n^2 - 18n + c) = g(n).$$

$g(n) = 0$ when

$$n = \frac{18 \pm \sqrt{324 - 12c}}{6} = n_{\pm}, \quad \text{say.}$$

2 real roots provided $c \leq 27$. $g'(n) < 0$ at $n = 0$ implies graph looks like this:

So

$$n_s = n_+ = \frac{18 + \sqrt{324 - 12c}}{6}$$

for stable equilibrium.

For $c > 27$ have no real roots, so graph looks like this:

and $n = 0$ is the only stable equilibrium; the gazelles die out.

Catch

$$\begin{aligned} C =& cn_s = c\frac{18 + \sqrt{324 - 12c}}{6}. \\ \frac{dC}{dc} =& \frac{18 + \sqrt{324 - 12c}}{6} + \tfrac{1}{2}(-12)\frac{(324 - 12c)^{-\frac{1}{2}}}{6} = 0. \\ \Rightarrow \frac{18 + \sqrt{324 - 12c}}{6} =& \frac{1}{\sqrt{324 - 12c}}c. \end{aligned}$$

satisfied by $c = 24$.

$$\frac{d^2C}{dc^2} = \tfrac{1}{2}(-12)\frac{(324-12c)^{-\frac{1}{2}}}{6} + \tfrac{1}{2}(-12)\frac{(324-12c)^{-\frac{1}{2}}}{6} + \tfrac{1}{2}$$
$$+ \tfrac{1}{2}(-12)\left(-\tfrac{1}{2}\right)(-12)\frac{(324-12c)^{-\frac{3}{2}}}{6}c < 0$$

So $c = 24$ is a maximum.

6.

$$\frac{dn}{dt} = 3n^2 - 7n + 4 = (n-1)(3n-4) = f(n).$$

The equilibrium densities are $n = 1$ and $n = \frac{4}{3}$. The graph of $f(n)$ looks like this:

so $n = 1$ is stable, $n = \frac{4}{3}$ unstable.

Writing

$$\begin{aligned}
\frac{1}{3n^2 - 7n + 4} =& \frac{A}{n-1} + \frac{B}{3n-4} \\
\Rightarrow 1 =& (3n-4)A + (n-1)B. \\
n = 1 \Rightarrow A = -1, \quad & n = \tfrac{4}{3} \Rightarrow B = 3 \\
\Rightarrow \int_0^n \left[\frac{3}{3n-4} - \frac{1}{n-1}\right] =& t \\
\Rightarrow \ln \frac{3n-4}{4(n-1)} =& t \\
\Rightarrow 3n - 4 =& 4(n-1)e^t \Rightarrow n = 4\frac{1-e^{-t}}{4-3e^{-t}}.
\end{aligned}$$

As $t \to \infty$, $n \to 1$.

If $n(0) = \frac{3}{2}$, we have

$$\begin{aligned}
\frac{1}{5}\int_{\frac{3}{2}}^n \left[\frac{3}{3n-4} - \frac{1}{n-1}\right] =& t \\
\Rightarrow \ln \frac{(3n-4)}{(n-1)} =& t \\
\Rightarrow (3n-4) =& (n-1)e^t \Rightarrow n = \frac{1-4e^{-t}}{1-3e^{-t}}.
\end{aligned}$$

As $t \to \ln 3$, $n \to \infty$ in line with the instability of $n = \frac{4}{3}$.

7. (a)

$$\frac{dn}{dt} = -14n + 9n^2 - n^3 = -n(n-2)(n-7) = f(n).$$

Equilibrium densities $n = 0$, $n = 2$, $n = 7$.

$f'(0) < 0$, $f(n) \to -\infty$ as $n \to \infty$. So graph looks like this:

Equilibria at $n = 0$, $n = 7$ stable; equilibrium at $n = 2$ unstable.

(b)

$$\frac{dx}{dt} = x(2-4x) + xy, \quad \frac{dy}{dt} = y(7-3y) - xy,$$

Terms (1), (3) are logistic growth functions, implying each population could survive on its own in a limited resource environment.

Terms (2), (4) with opposite signs imply the first species is preying on the second.

$$\begin{aligned} \text{Either} \quad 2 - 4x + y &= 7 - x - 3y = 0 \Rightarrow x = 1, \quad y = 2 \\ \text{or} \quad x = 7 - x - 3y &= 0 \Rightarrow y = \tfrac{7}{3}, \\ \text{or} \quad y = 2 - 4x + y &= 0 \Rightarrow x = \tfrac{1}{2}, \\ \text{or} \quad x = y = 0. \end{aligned}$$

So the equilibria are $(0,0)$, $(0, \frac{7}{3})$, $(\frac{1}{2}, 0)$, $(1, 2)$.

Community matrix

$$A = \begin{pmatrix} (2 - 4x + y) - 4x & x \\ -y & (7 - x - 3y) - 3y \end{pmatrix}.$$

For $(0,0)$, $A = \begin{pmatrix} 2 & 0 \\ 0 & 7 \end{pmatrix}$. E-values 2, 7 both positive $\Rightarrow$ improper node, unstable.

For $(0, \frac{7}{3})$, $A = \begin{pmatrix} \frac{13}{3} & 0 \\ -\frac{7}{3} & -7 \end{pmatrix}$. E-values $\frac{13}{3}$, -7 opposite signs $\Rightarrow$ saddle point.

For $(\frac{1}{2}, 0)$, $A = \begin{pmatrix} -2 & \frac{1}{2} \\ 0 & \frac{13}{2} \end{pmatrix}$. E-values -2, $\frac{13}{2}$ opposite signs $\Rightarrow$ saddle point.

For $(1, 2)$, $A = \begin{pmatrix} -4 & 1 \\ -2 & -6 \end{pmatrix}$.

Linearised equations

$$\begin{aligned}
\frac{d\epsilon_x}{dt} =& -4\epsilon_x + \epsilon_y \\
\frac{d\epsilon_y}{dt} =& -2\epsilon_x - 6\epsilon_y. \\
-4\epsilon_x + \epsilon_y =& e^{-5t}\left[-4\delta\cos t - \delta(\cos t + \sin t)\right] \\
=& e^{-5t}\left[-5\delta\cos t - \delta\sin t\right] = \frac{d\epsilon_x}{dt} \\
-2\epsilon_x - 6\epsilon_y =& e^{-5t}\left[-2\delta\cos t + 6\delta(\cos t + \sin t)\right] \\
=& e^{-5t}\left[4\delta\cos t + 6\delta\sin t\right] = \frac{d\epsilon_y}{dt}
\end{aligned}$$

Also $\epsilon_x(0) = \delta$, $\epsilon_y(0) = -\delta$.