MATH727 MATHEMATICAL METHODS FOR NON-PHYSICAL SYSTEMS

JANUARY 2006

Full marks can be obtained for complete answers to FIVE questions. Only the best FIVE answers will be counted.

1. Pat's preferences for tea and coffee are quantified by the utility function

$$U(x, y) = xy + 3x + 5y + 2,$$

where x and y denote the numbers of cups of tea and cups of coffee drunk per week, respectively.

Show that Pat prefers N cups of coffee, with no tea, to N cups of tea, with no coffee.

Express the indifference curve $U(x, y) = U_0$ in the explicit form

$$y = f(x, U_0),$$

for some function f, and sketch a selection of indifference curves for various values of U_0.

Verify that

$$\frac{df}{dx} < 0 \quad \text{and} \quad \frac{d^2 f}{dx^2} > 0,$$

within the domain of U. Comment on the graphical significance of these inequalities.

Tea costs £p per cup and coffee costs £q per cup. Pat has £12 per week to spend on these drinks. Show that

$$x(p, q) = \frac{12 + 3q}{2p} - \frac{5}{2}.$$

Find a corresponding expression for $y(p, q)$.

The own-elasticity of Pat's demand for tea, ϵ_x, is defined by

$$\epsilon_x = \frac{p}{x}\frac{dx}{dp},$$

where q is regarded as a constant. Compute ϵ_x and show that $\epsilon_x < -1$ if $5p - 3q < 12$.

Compute also the own-elasticity of Pat's demand for coffee, ϵ_y, defined by

$$\epsilon_y = \frac{q}{y}\frac{dy}{dq},$$

where now p is regarded as a constant.

Sketch the demand curve $x(p, 1)$.

[20 marks]

2. (a) A firm has production function

$$q(x,y) = (x+1)^{\frac{1}{3}}(y+2)^{\frac{2}{3}}$$

and cost function

$$c(x,y) = 27x + 2y + 4$$

per unit time, with inputs x and y. Find the minimum value of $c(x,y)$ consistent with a fixed production level of 12 units per unit time.

[6 marks]

(b) The total cost function for a single-commodity firm is

$$C(q) = q^3 - 8q^2 + 24q + 12,$$

where q is the quantity of the commodity produced in unit time.

Determine:

(i) The fixed cost;

(ii) The marginal cost function, $MC(q)$;

(iii) The average variable cost function, $AVC(q)$.

Find the price at which the firm would be forced to cease production in the *short-run* in an ideal competitive market.

[6 marks]

(c) A monopolist has cost function

$$C(q) = q^3 - 6q^2 + 14q + 3,$$

for the production of a quantity q of a commodity, per unit time. Write down the profit function, as a function of q, when the demand function is

$$D(p) = 22 - p, \quad p < 22.$$

Determine the market price. Verify that your answer maximises the monopolist's profit function for $q \geq 0$.

[8 marks]

3. Each of N identical firms producing a single commodity has cost function

$$C(q) = q^3 - 4q^2 + 6q + 64.$$

Show that each firm has supply function

$$S(p) = \begin{cases} \frac{4+\sqrt{3p-2}}{3} & \text{if } p \geq 2 \\ 0 & \text{if } p < 2. \end{cases}$$

Find the equilibrium price and the profit/loss made by each firm in an ideal competitive market where the demand function is

$$D(p) = N(3 - \frac{1}{2}p).$$

Find the price below which production is not viable in the *long-run.*

A monopolist takes over all of the firms. If the total cost of producing Nq goods is $NC(q)$, find the new equilibrium price.

[20 marks]

4. Two firms producing the same commodity have cost functions

$$C_1(q_1) = 5 + 4q_1 + q_1^2,$$
$$C_2(q_2) = 7 + 9q_2 + \tfrac{1}{2}q_2^2,$$

where q_1 and q_2 are the respective quantities of production per unit time.

The two firms form a Cournot duopoly to supply a market which has demand function $D(p) = 14 - p$. Find the production of each firm. Find the price charged per unit of the good and the profits made by the companies.

The firms tentatively agree to co-operate and maximise their joint profit. Find what the individual profits would be, if such an agreement were carried out.

The second firm now publicly announces that it will raise and maintain its production at $q_2 = 4$ units per unit time. Determine the response of the first company.

Does the second company benefit by its unilateral action?

[20 marks]

5. (a) The community matrix at a particular equilibrium point of a two-species model has distinct negative eigenvalues. Describe briefly, with the aid of diagrams, the possible local behaviour of trajectories near to the equilibrium point.

[5 marks]

(b) The population density of gazelles, n, in a well-defined region of a wildlife reserve satisfies

$$\frac{dn}{dt} = 18n^2 - 3n^3.$$

Find the non-zero equilibrium density, and determine its stability.

A pride of lions starts preying upon the gazelles. The growth law for the population density of gazelles becomes modified:

$$\frac{dn}{dt} = 18n^2 - 3n^3 - cn,$$

where c is the number of lions hunting gazelles each day. Given that $c \leq 27$, find the new equilibrium population densities. Identify the non-zero density n_s at which the population is in stable equilibrium.

Explain what would happen if more than 27 lions hunted gazelles each day.

Assuming $c \leq 27$, and the number of gazelles caught each day by each lion is proportional to n_s, find the value of c which maximises the total number caught daily by the pride.

[15 marks]

6. The population density of a particular species satisfies the growth equation

$$\frac{dn}{dt} = 3n^2 - 7n + 4.$$

Find the equilibrium densities, and determine their stability.

Express the function

$$\frac{1}{3n^2 - 7n + 4}$$

in partial fractions, and hence obtain the solution of the growth equation with initial condition $n(0) = 0$. Describe what happens to the population density as $t \to \infty$.

Find also the solution of the growth equation with $n(0) = \frac{3}{2}$, for $t < \ln 3$. Comment on what happens as t approaches $\ln 3$.

[20 marks]

7. (a) The population density, $n(t)$, of a species of fish satisfies

$$\frac{dn}{dt} = -20n + 9n^2 - n^3.$$

Find the equilibrium densities, and determine their stability.

[4 marks]

(b) The population densities of two interacting species evolve with time according to

$$\frac{dx}{dt} = x(2-4x) + xy, \quad \frac{dy}{dt} = y(7-3y) - xy,$$

where $x(t)$ and $y(t)$ are the population densities of the two species.

Comment on the biological significance of each of the terms on the right hand sides of these equations.

Find the equilibria and classify all except the coexistence equilibrium.

Let the coexistence equilibrium be (x_c, y_c). Write down the linearised growth equations in the neighbourhood of (x_c, y_c), using the substitutions $x = x_c + \epsilon_x$, $y = y_c + \epsilon_y$ where the deviations from equilibrium are small. Verify that the particular solutions corresponding to the initial conditions $x(0) = x_c + \delta$, $y(0) = y_c - \delta$ are given approximately by

$$x(t) = x_c + \delta e^{-5t} \cos t, \quad y(t) = y_c - \delta e^{-5t} [\cos t + \sin t].$$

What kind of equilibrium is this?

[16 marks]