MATH727 MATHEMATICAL METHODS FOR NON-PHYSICAL SYSTEMS

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1.

$$U(0,N) = 4(N+1) > U(N,0) = N+4$$

so N bars of chocolate preferred to N bags of crisps.

Indifference curves given by

$$y = \frac{U_0+2}{x+4} - 1$$

$$\frac{dy}{dx} = -\frac{U_0+2}{(x+4)^2} < 0,$$

$$\frac{d^2y}{dx^2} = 2\frac{U_0+2}{(x+4)^3} > 0,$$

Budget constraint touches indifference curve where

$$\frac{\frac{\partial U}{\partial x}}{\frac{\partial U}{\partial y}} = \frac{p}{1}$$

$$\frac{y+1}{x+4} = p \Rightarrow y = px + 4p - 1.$$

Substituting into budget constraint $px + y = 7$ we have

$$2px + 4p = 8 \Rightarrow x(p) = \frac{4-2p}{p} = \frac{4}{p} - 2 \Rightarrow y = 3 + 2p.$$

So buys $\frac{4}{p} - 2$ bags of crisps, $3 + 2p$ bars of chocolate.

$$\epsilon_x = p\frac{p}{4-2p}\left(-\frac{4}{p^2}\right) = \frac{2}{p-2}$$
$$\Rightarrow \epsilon_x + 1 = \frac{p}{p-2} < 0 \quad \text{if} \quad p < 2.$$

So $\epsilon_x < -1$ if $p < 2$.

2. (a) $c(x,y)$ is minimised where

$$\frac{\frac{\partial q}{\partial x}}{\frac{\partial q}{\partial y}} = \frac{\frac{\partial c}{\partial x}}{\frac{\partial c}{\partial y}}$$

$$\frac{\frac{4}{5}(x+1)^{-\frac{1}{5}}y^{\frac{1}{5}}}{\frac{1}{5}(x+1)^{\frac{4}{5}}y^{-\frac{4}{5}}} = \frac{4}{3}$$

$$4\frac{y}{(x+1)} = \frac{4}{3} \Rightarrow y = \tfrac{1}{3}(x+1)$$

This is the expansion path.

$$\Rightarrow \left(\tfrac{1}{3}\right)^{\frac{1}{5}}(x+1) = 3 \Rightarrow x = 3^{\frac{6}{5}} - 1, \quad y = 3^{\frac{1}{5}}$$

$$\Rightarrow c = 4x + 3y + 2 = 4(3^{\frac{6}{5}} - 1) + 3.3^{\frac{1}{5}} + 2 = 5.3^{\frac{6}{5}} - 2.$$

Minimum cost for production level of 3 units is $5.3^{\frac{6}{5}} - 2$ units.

(b)

$$C(q) = q^3 - 6q^2 + 14q + 10.$$

(i) Fixed cost $C(0) = 10$.

(ii) $MC(q) = C'(q) = 3q^2 - 12q + 14$.

(iii) $AVC(q) = \frac{C(q)-C(0)}{q} = q^2 - 6q + 14$.

Cease production when $p = \min(AVC)$.

$$AVC'(q) = 2q - 6 = 0 \quad \text{when} \quad q = 3 \Rightarrow p = AVC(3) = 5.$$

(c)

$$C(q) = q^3 - 5q^2 + 9q + 3, \quad D(p) = 12 - p = q \Rightarrow p = 12 - q.$$

Profit given by

$$P(q) = pq - C(q) = (12-q)q - (q^3 - 5q^2 + 9q + 3) = -\,q^3 + 4q^2 + 3q - 3$$

$$\Rightarrow P'(q) = -3q^2 + 8q + 3 = -(3q+1)(q-3) = 0 \quad \text{for} \quad q = -\tfrac{1}{3}, \quad 3.$$

Take the +ve solution $q = 3$. Then $p = 12 - q = 9$.

$$P''(q) = -6q + 8 < 0 \quad \text{for} \quad q = 3.$$

So we have a local maximum. Also

$$P(3) = -3^3 + 4.3^2 + 3.3 - 3 = 15 > P(0) = -3.$$

So $q = 3$ is a global maximum.

3.

$$C(q) = q^3 - 2q^2 + 4q + 36 \Rightarrow AVC(q) = q^2 - 2q + 4.$$
$$AVC'(q) = 2q - 2 = 0 \quad \text{when} \quad q = 1 \Rightarrow \min(AVC) = 3.$$

So cease production when $p = \min(AVC) = 3$. For $p \geq 3$,

$$p = C'(q) = 3q^2 - 4q + 4 \Rightarrow 3q^2 - 4q + 4 - p = 0$$
$$\Rightarrow q = \frac{4 \pm \sqrt{16 - 12(4-p)}}{6} = \frac{2 \pm \sqrt{3p-8}}{3}.$$

Take +ve sign for maximum profit. So

$$S(p) = \begin{cases} \frac{2+\sqrt{3p-8}}{3} & \text{if } p \geq 3 \\ 0 & \text{if } p < 3. \end{cases}$$

Equilibrium is when $NS(p) = D(p)$ (N firms) so

$$\frac{2 + \sqrt{3p-8}}{3} = 4 - p.$$

$p = 3$ is a solution by inspection, and since $D(p)$ is decreasing and $S(p)$ is increasing, it is unique.

$$p = 3 \Rightarrow q = \frac{1}{N} D(p) = 1 \Rightarrow P(q) = pq - C(q)$$
$$= 3 - (1 - 2 + 4 + 36) = -36.$$

So each firm makes a loss of 36 units.

Production not viable in the long-run for $p < \min(ATC)$.

$$ATC = q^2 - 2q + 4 + \frac{36}{q} \Rightarrow ATC'(q) = 2q - 2 - \frac{36}{q^2}$$
$$= 0 \quad \text{when} \quad q = 3,$$

by inspection. It is a minimum, since

$$ATC''(q) = 2 + \frac{72}{q^3} > 0.$$
$$\min(ATC) = 9 - 6 + 4 + 12 = 19.$$

So minimum price in the long-run is 19 units.

In monopoly case

$$P(q) = N[pq - C(q)] = N[(4-q)q - (q^3 - 2q^2 + 4q + 36)] = N[-q^3 + q^2 - 36].$$
$$\Rightarrow P'(q) = N[-3q^2 + 2q] = 0 \quad \text{when} \quad q = 0, \quad \tfrac{2}{3}.$$

Take $q = \frac{2}{3} \Rightarrow p = 4 - \frac{2}{3} = \frac{10}{3}$.

4.

$$\begin{aligned} C_1(q_1) =& 6 + 7q_1 + \frac{1}{2}q_1^2, \\ C_2(q_2) =& 4 + 8q_2 + q_2^2, \end{aligned}$$

Profits:

$$\begin{aligned} P_1(q_1,q_2) =& pq_1 - (6 + 7q_1 + \frac{1}{2}q_1^2) = [14 - (q_1+q_2)]q_1 - (6 + 7q_1 + \frac{1}{2}q_1^2) \\ =& -\frac{3}{2}q_1^2 - q_1q_2 + 7q_1 - 6, \\ P_2(q_1,q_2) =& pq_2 - (4 + 8q_2 + q_2^2) = [14 - (q_1+q_2)]q_2 - (4 + 8q_2 + q_2^2) \\ =& -q_1q_2 - 2q_2^2 + 6q_2 - 4. \end{aligned}$$

Cournot duopoly $\Rightarrow$ maximise P_1, P_2 wrto q_1, q_2 respectively. So

$$\begin{aligned} \frac{\partial P_1}{\partial q_1} =& -3q_1 - q_2 + 7 = 0, \\ \frac{\partial P_2}{\partial q_2} =& -q_1 - 4q_2 + 6 = 0, \end{aligned}$$

Then $q_1 = 2$, $q_2 = 1$. So $p = 14 - 2 - 1 = 11$ and $P_1(2,1) = 0$, $P_2(2,1) = -2$.

If co-operate, maximise

$$\begin{aligned} P(q_1,q_2) =& P_1(q_1,q_2) + P_2(q_1,q_2) \\ =& -\frac{3}{2}q_1^2 - 2q_1q_2 + 7q_1 - 2q_2^2 + 6q_2 - 10 \\ \frac{\partial P_1}{\partial q_1} =& -3q_1 - 2q_2 + 7 = 0, \\ \frac{\partial P_2}{\partial q_2} =& -2q_1 - 4q_2 + 6 = 0, \end{aligned}$$

giving $q_1 = 2$, $q_2 = \frac{1}{2}$. Then $P_1(2, \frac{1}{2}) = 1$, $P_2(2, \frac{1}{2}) = -\frac{5}{2}$.

If $q_2 = 2$, have

$$P_1(q_1, 2) = -\frac{3}{2}q_1^2 - 2q_1 + 7q_1 - 6 = -\frac{3}{2}q_1^2 + 5q_1 - 6,$$

$$\frac{\partial P_1}{\partial q_1} = -3q_1 + 5 = 0 \quad \text{when} \quad q_1 = \tfrac{5}{3}.$$

So the first firm lowers its production to $q_1 = \frac{5}{3}$ units.

Then

$$P_2(\tfrac{5}{3}, 2) = -q_1q_2 - 2q_2^2 + 6q_2 - 4 = -\tfrac{5}{3}.2 - 2.2^2 + 6.2 - 4 = -\tfrac{10}{3}$$

so the second firm doesn't benefit.

5. (a)

$$\mathbf{x} = \mathbf{x}^e + c_1\mathbf{x}_1 e^{\lambda_1 t} + c_2\mathbf{x}_2 e^{\lambda_2 t},$$

where $\lambda_{1,2}$ are the e-values, $\mathbf{x}_{1,2}$ are the e-vectors. If λ_1 and λ_2 have opposite signs, then we have a saddle point; the trajectories always move away from $\mathbf{x}_e$.

(b) τ is the low-density relaxation time (i.e. the time for the population density to be multiplied by e). N is the maximum sustainable population density.

$$\begin{aligned}
\frac{dn}{dt} &= \frac{n}{\tau}\left(1 - \frac{n}{N}\right), \\
\frac{1}{n\left(1 - \frac{n}{N}\right)} &= \frac{1}{n} + \frac{1}{N-n} \\
\int_{n_0}^{n} dn \left(\frac{1}{n} + \frac{1}{N-n}\right) &= \int_0^t \frac{dt}{\tau} \\
\left[\ln\left(\frac{n}{N-n}\right)\right] &= \frac{t}{\tau} \\
\frac{n}{n_0}\frac{N-n_0}{N-n} &= e^{\frac{t}{\tau}} \\
n[N - n_0 + n_0 e^{\frac{t}{\tau}}] = n_0 N e^{\frac{t}{\tau}} & \\
n &= \frac{n_0 N}{n_0 + (N-n_0)e^{\frac{-t}{\tau}}}.
\end{aligned}$$

When $t = \tau$, $n = 2n_0$, so

$$\begin{aligned}
2n_0 =& \frac{n_0 N}{n_0 + (N - n_0)e^{-1}} \\
\Rightarrow 2en_0 + 2(N - n_0) =& Ne \\
N(e-2) =& 2(e-1)n_0 \\
N =& \frac{2(e-1)n_0}{e-2}. \\
n(2\tau) =& \frac{n_0 N}{n_0 + (N - n_0)e^{-2}} \\
N - n_0 =& \frac{en_0}{e-2} \\
\Rightarrow n(2\tau) =& \frac{n_0 \frac{2(e-1)n_0}{e-2}}{n_0 + \frac{en_0}{e-2} e^{-2}} \\
=& \frac{2(e-1)n_0}{(e-2) + \frac{1}{e}} \approx 3.2n_0.
\end{aligned}$$

6.

$$\frac{dn}{dt} = 2n^2 - 7n + 3 = (2n-1)(n-3) = f(n).$$

The equilibrium densities are $n = \frac{1}{2}$ and $n = 3$. The graph of $f(n)$ looks like this:

so $n = \frac{1}{2}$ is stable, $n = 3$ unstable.

Writing

$$\begin{aligned}
\frac{1}{2n^2 - 7n + 3} =& \frac{A}{n-3} + \frac{B}{2n-1} \\
\Rightarrow 1 =& (2n-1)A + (n-3)B.
\end{aligned}$$

$$n = \tfrac{1}{2} \Rightarrow B = -\tfrac{2}{5}, \quad n = 3 \Rightarrow A = \tfrac{1}{5}$$

$$\begin{aligned}
\Rightarrow \frac{1}{5}\int_0^n \left[\frac{1}{n-3} - \frac{2}{2n-1}\right] =& t \\
\Rightarrow \frac{1}{5}\ln\frac{n-3}{3(2n-1)} =& t \\
\Rightarrow n - 3 =& 3(2n-1)e^{5t} \Rightarrow n = 3\frac{1-e^{-5t}}{6-e^{-5t}}.
\end{aligned}$$

As $t \to \infty$, $n \to \frac{1}{2}$.

If $n(0) = 8$, we have

$$\begin{aligned}
\frac{1}{5}\int_8^n \left[\frac{1}{n-3} - \frac{2}{2n-1}\right] =& t \\
\Rightarrow \frac{1}{5}\ln\frac{3(n-3)}{2n-1} =& t \\
\Rightarrow 3(n-3) =& (2n-1)e^{5t} \Rightarrow n = \frac{1-9e^{-5t}}{2-3e^{-5t}}.
\end{aligned}$$

7. (a)

$$\frac{dn}{dt} = -14n + 9n^2 - n^3 = -n(n-2)(n-7) = f(n).$$

Equilibrium densities $n = 0$, $n = 2$, $n = 7$.

$f'(0) < 0$, $f(n) \to -\infty$ as $n \to \infty$. So graph looks like this:

Equilibria at $n = 0$, $n = 7$ stable; equilibrium at $n = 2$ unstable.

(b)

$$\frac{dx}{dt} = x(5-x) - xy, \quad \frac{dy}{dt} = y(7-2y) - xy,$$

Terms (1), (3) are logistic growth functions, implying each population could survive on its own in a limited resource environment.

Terms (2), (4) both with negative signs imply competiton between the two species.

$$\begin{aligned} \text{Either} \quad 5 - x - y =& 7 - x - 2y = 0 \Rightarrow x = 3, \quad y = 2 \\ \text{or} \quad x = 7 - x - 2y =& 0 \Rightarrow y = \tfrac{7}{2}, \\ \text{or} \quad y = 5 - x - y =& 0 \Rightarrow x = 5, \\ \text{or} \quad x = y = 0. \end{aligned}$$

So the equilibria are $(0,0)$, $(0, \frac{7}{2})$, $(5,0)$, $(3,2)$.

Community matrix

$$A = \begin{pmatrix} (5-x-y) - x & -x \\ -y & (7-x-2y) - 2y \end{pmatrix}.$$

For $(0,0)$, $A = \begin{pmatrix} 5 & 0 \\ 0 & 7 \end{pmatrix}$. E-values 5, 7 both positive $\Rightarrow$ improper node, unstable.

For $(0, \frac{7}{2})$, $A = \begin{pmatrix} \frac{3}{2} & 0 \\ -\frac{7}{2} & -7 \end{pmatrix}$. E-values $\frac{3}{2}$, -7 opposite signs $\Rightarrow$ saddle point.

For $(5,0)$, $A = \begin{pmatrix} -5 & -5 \\ 0 & 2 \end{pmatrix}$. E-values -5, 2 opposite signs $\Rightarrow$ saddle point.

For $(3,2)$, $A = \begin{pmatrix} -3 & -3 \\ -2 & -4 \end{pmatrix}$.

Linearised equations

$$\begin{aligned}
\frac{d\epsilon_x}{dt} &= -\,3\epsilon_x - 3\epsilon_y \\
\frac{d\epsilon_y}{dt} &= -\,2\epsilon_x - 4\epsilon_y. \\
-3\epsilon_x - 3\epsilon_y &= -\,\tfrac{3}{5}\delta\left[3e^{-t} + 2e^{-6t}\right] - \tfrac{3}{5}\delta\left[-2e^{-t} + 2e^{-6t}\right] \\
&= -\,\tfrac{3}{5}\delta e^{-t} - \tfrac{12}{5}\delta e^{-6t} = \frac{d\epsilon_x}{dt} \\
-2\epsilon_x - 4\epsilon_y &= -\,\tfrac{2}{5}\delta\left[3e^{-t} + 2e^{-6t}\right] - \tfrac{4}{5}\delta\left[-2e^{-t} + 2e^{-6t}\right] \\
&= \tfrac{2}{5}\delta e^{-t} - \tfrac{12}{5}\delta e^{-6t} = \frac{d\epsilon_y}{dt}
\end{aligned}$$

Also $\epsilon_x(0) = 1$, $\epsilon_y(0) = 0$.