MATH727 MATHEMATICAL METHODS FOR NON-PHYSICAL SYSTEMS

JANUARY 2005

Full marks can be obtained for complete answers to FIVE questions. Only the best FIVE answers will be counted.

1. Sam's preferences for crisps and chocolate are quantified by the utility function

$$U(x, y) = (x + 4)(y + 1) - 2,$$

where x and y denote the numbers of packets of crisps and bars of chocolate eaten per week, respectively.

Show that Sam prefers N bars of chocolate, with no crisps, to N bags of crisps, with no chocolate.

Express the indifference curve $U(x, y) = U_0$ in the explicit form

$$y = f(x, U_0),$$

for some function f, and sketch a selection of indifference curves for various values of U_0.

Verify that

$$\frac{df}{dx} < 0 \quad \text{and} \quad \frac{d^2 f}{dx^2} > 0,$$

within the domain of U. Comment on the graphical significance of these inequalities.

Crisps cost £p per bag and chocolate costs £1 per bar. Sam has £7 per week to spend on these snacks. How many bags of crisps and bars of chocolate does Sam buy per week, as a function of p? Sketch the demand curve $x(p)$ for $0 < p \leq 2$.

The own-elasticity of Sam's demand for crisps, ϵ_x, is defined by

$$\epsilon_x = \frac{p}{x}\frac{dx}{dp}.$$

Compute ϵ_x and show that $\epsilon_x < -1$ if $p < 2$.

[20 marks]

2. (a) A firm has production function

$$q(x, y) = (x + 1)^{\frac{4}{5}} y^{\frac{1}{5}}$$

and cost function

$$c(x, y) = 4x + 3y + 2$$

per unit time, with inputs x and y. Find the equation of the expansion path. Also find the minimum value of $c(x, y)$ consistent with a fixed production level of 3 units per unit time.

[6 marks]

(b) The total cost function for a single-commodity firm is

$$C(q) = q^3 - 6q^2 + 14q + 10,$$

where q is the quantity of the commodity produced in unit time.

Determine:

(i) The fixed cost;

(ii) The marginal cost function, $MC(q)$;

(iii) The average variable cost function, $AVC(q)$.

Find the price at which the firm would be forced to cease production in the *short-run* in an ideal competitive market.

[6 marks]

(c) A monopolist has cost function

$$C(q) = q^3 - 5q^2 + 9q + 3,$$

for the production of a quantity q of a commodity, per unit time. Write down the profit function, as a function of q, when the demand function is

$$D(p) = 12 - p, \quad p < 12.$$

Determine the market price, and the monopolist's profit. Verify that your answer maximises the monopolist's profit function for $q \geq 0$.

[8 marks]

3. Each of N identical firms producing a single commodity has cost function

$$C(q) = q^3 - 2q^2 + 4q + 36.$$

Show that each firm has supply function

$$S(p) = \begin{cases} \frac{2+\sqrt{3p-8}}{3} & \text{if } p \geq 3 \\ 0 & \text{if } p < 3. \end{cases}$$

Find the equilibrium price and the profit/loss made by each firm in an ideal competitive market where the demand function is

$$D(p) = N(4 - p).$$

Find the price below which production is not viable in the *long-run*.

A monopolist takes over all of the firms. If the total cost of producing Nq goods is $NC(q)$, find the new equilibrium price. (You may assume that the global maximum occurs for $q \neq 0$.)

[20 marks]

4. Two firms producing the same commodity have cost functions

$$C_1(q_1) = 6 + 7q_1 + \frac{1}{2}q_1^2,$$
$$C_2(q_2) = 4 + 8q_2 + q_2^2,$$

where q_1 and q_2 are the respective quantities of production per unit time.

The two firms form a Cournot duopoly to supply a market which has demand function $D(p) = 14 - p$. Find the production of each firm. Find the price charged per unit of the good and the profits made by the companies.

The firms tentatively agree to co-operate and maximise their joint profit. Find what the individual profits would be, if such an agreement were carried out.

The second firm now publicly announces that will raise and maintain its production at $q_2 = 2$ units per unit time. Determine the response of the first company.

Does the second company benefit by its unilateral action?

[20 marks]

5. (a) The community matrix at a particular equilibrium point of a two-species model has distinct eigenvalues with opposite signs. Describe briefly, with the aid of diagrams, the possible local behaviour of trajectories near to the equilibrium point.

[5 marks]

(b) The population density $n(t)$ of a certain species evolves according to the logistic equation

$$\frac{dn}{dt} = \frac{n}{\tau}\left(1 - \frac{n}{N}\right),$$

where N and τ are positive constants. Explain the biological significance of these constants.

Solve the logistic equation given that the initial condition $n(0) = n_0 < N$ is satisfied.

At $t = \tau$ the population is found to be twice its initial value. Deduce that

$$N = \frac{2(e-1)}{(e-2)} n_0,$$

and show that

$$n(2\tau) \approx 3.2 n_0.$$

[15 marks]

6. The population density of a particular species satisfies the growth equation

$$\frac{dn}{dt} = 2n^2 - 7n + 3.$$

Find the equilibrium densities, and determine their stability.

Express the function

$$\frac{1}{2n^2 - 7n + 3}$$

in partial fractions, and hence obtain the solution of the growth equation with initial condition $n(0) = 0$. Describe what happens to the population density as $t \to \infty$.

Find also the solution of the growth equation with $n(0) = 8$.

[20 marks]

7. (a) The population density, $n(t)$, of a species of fish satisfies

$$\frac{dn}{dt} = -14n + 9n^2 - n^3.$$

Find the equilibrium densities, and determine their stability.

[6 marks]

(b) The population densities of two interacting species evolve with time according to

$$\frac{dx}{dt} = x(5-x) - xy, \quad \frac{dy}{dt} = y(7-2y) - xy,$$

where $x(t)$ and $y(t)$ are the population densities of the two species.

Comment on the biological significance of each of the terms on the right hand sides of these equations.

Show that there are equilibria at $(0,0)$, $(0,\frac{7}{2})$, $(5,0)$, $(3,2)$ and classify the first two of them.

Write down the linearised growth equations in the neighbourhood of the coexistence equilibrium, $(3,2)$, using the substitutions $x = 3+\epsilon_x$, $y = 2+\epsilon_y$ where the deviations from equilibrium are small. Verify that the particular solutions corresponding to the initial conditions $x(0) = 3+\delta$, $y(0) = 2$ are given approximately by

$$x(t) = 3 + \tfrac{1}{5}\delta\left[3e^{-t} + 2e^{-6t}\right], \quad y(t) = 2 + \tfrac{1}{5}\delta\left[-2e^{-t} + 2e^{-6t}\right].$$

[14 marks]