MATH727 MATHEMATICAL METHODS FOR NON-PHYSICAL SYSTEMS

JANUARY 2004

Full marks can be obtained for complete answers to FIVE questions. Only the best FIVE answers will be counted.

1. Bev's preferences for tea and coffee are quantified by the utility function

$$U(x, y) = (xy^2 + 1)^2,$$

where x and y denote the numbers of cups of tea and coffee drunk per day, respectively.

Sketch a selection of indifference curves for U, and verify that the conditions for U to be a utility function are satisfied.

Bev is on holiday in France, where tea costs 3 euros per cup and coffee costs 4 euros per cup. Bev has 18 euros per day to spend on drinks. How many cups of each does Bev drink per day?

Bev now travels to Germany where the price of tea is the same but coffee now costs 6 euros per cup. How do her drinking habits change?

On her last day in Germany she treats herself to a large cream cake costing 9 euros which she pays for out of her drinks budget. How many cups of coffee and tea does she drink that day?

Finally she goes to Italy, where she stays several weeks (still on her daily drinks budget of 18 euros). The cost of tea is a variable p euros per cup, while coffee is 4 euros per cup. How many cups of tea and coffee does she now drink, as a function of p?

[20 marks]

2. (i) A firm has production function

$$q(x, y) = x^{\frac{1}{4}} y^{\frac{3}{4}}$$

and cost function

$$c(x, y) = 3x + 4y + 10$$

per unit time, with inputs x and y. Find the equation of the expansion path. Also find the minimum value of $c(x, y)$ consistent with a fixed production level of 3 units per unit time.

[6 marks]

(ii) The total cost function for a single-commodity firm is

$$C(q) = q^3 - 8q^2 + 20q + 8,$$

where q is the quantity of the commodity produced in unit time.

Determine:

(a) The fixed cost;

(b) The marginal cost function, $MC(q)$;

(c) The average variable cost function, $AVC(q)$.

Find the price at which the firm would be forced to cease production in the *short-run* in an ideal competitive market.

[6 marks]

(iii) A monopolist has cost function

$$C(q) = q^3 - 3q^2 + 5q + 4,$$

for the production of a quantity q of a commodity, per unit time. Write down the profit function, as a function of q, when the demand function is

$$D(p) = 20 - p, \quad p < 20.$$

Determine the market price, and the monopolist's profit. Verify that this is maximises the monopolist's profit for $q \geq 0$.

[8 marks]

3. Each of N identical firms producing a single commodity has cost function

$$C(q) = q^3 - 4q^2 + 8q + 18.$$

Show that each firm has supply function

$$S(p) = \begin{cases} \frac{4+\sqrt{3p-8}}{3} & \text{if } p \geq 4 \\ 0 & \text{if } p < 4. \end{cases}$$

Find the equilibrium price and the profit/loss made by each firm in an ideal competitive market where the demand function is

$$D(p) = N(4 - \frac{1}{2}p).$$

Find the price below which production is not viable in the *long-run.*

A monopolist takes over all of the firms. If the total cost of producing Nq goods is $NC(q)$, find the new equilibrium price. (You may assume that the global maximum occurs for $q \neq 0$.)

[20 marks]

4. Two firms producing the same commodity have cost functions

$$\begin{aligned} C_1(q_1) =& 8 + 5q_1 + q_1^2, \\ C_2(q_2) =& 6 + 4q_2 + \frac{3}{2}q_2^2, \end{aligned}$$

where q_1 and q_2 are the respective quantities of production per unit time.

The two firms form a Cournot duopoly to supply a market which has demand function $D(p) = 10 - p$. Find the production of each firm. Find the price charged per unit of the good and the profits made by the companies.

The firms tentatively agree to co-operate and maximise their joint profit. Find what the individual profits would be, if such an agreement were carried out.

However, the second firm decides to trade elsewhere, leaving the first as a monopoly. What is its profit now? Show that the latter of the three scenarios results in the highest price.

Suppose instead the first company had decided to trade elsewhere, leaving the second as a monopoly. What would the price have been then?

[20 marks]

5. (a) The community matrix at a particular equilibrium point of a two-species model has distinct eigenvalues of the same sign. Describe briefly, with the aid of diagrams, the possible local behaviour of trajectories near to the equilibrium point.

[6 marks]

(b) The population density $n(t)$ of a certain species evolves according to the logistic equation

$$\frac{dn}{dt} = \frac{n}{\tau}\left(1 - \frac{n}{N}\right),$$

where N and τ are positive constants. Explain the biological significance of these constants.

Solve the logistic equation given that the initial condition $n(0) = n_0 < N$ is satisfied.

At $t = \tau$ the population is found to be twice its initial value. Deduce that

$$N = \frac{2(e-1)}{(e-2)} n_0,$$

and show that

$$n(2\tau) \approx 3.2 n_0.$$

[14 marks]

6. The population density of gazelles, n, in a well-defined region of a wildlife reserve satisfies

$$\frac{dn}{dt} = 15n^2 - 5n^3.$$

Find the non-zero equilibrium density, and determine its stability.

A pride of lions starts preying upon the gazelles. The growth law for the population density of gazelles becomes modified:

$$\frac{dn}{dt} = 15n^2 - 5n^3 - cn,$$

where c is the number of lions hunting gazelles each day. Given that $c \leq 11$, find the new equilibrium population densities. Identify the density n_s at which the population is in stable equilibrium.

Explain what would happen if more than 11 lions hunted gazelles each day.

Assuming $c \leq 11$, and the number of gazelles caught each day by each lion is proportional to n_s, show that $c = 10$ maximises the daily catch.

If $c = 10$, and given that at $t = 0$, $n = 4$, show that $n(t)$ satisfies

$$\frac{3\sqrt{n(n-2)}}{2\sqrt{2}(n-1)} = e^{-5t}.$$

[20 marks]

7. The population densities of two interacting species evolve with time according to

$$\frac{dx}{dt} = x(6 - 4x) - 2xy, \qquad \frac{dy}{dt} = y(1 - 2y) + xy,$$

where $x(t)$ and $y(t)$ are the population densities of the two species.

Comment on the biological significance of each of the terms on the right hand sides of these equations.

Show that there are equilibria at $(0, 0)$, $(0, \frac{1}{2})$, $(\frac{3}{2}, 0)$, $(1, 1)$ and classify them.

Show that the general solution to the linearised equations may be written in the form

$$\begin{aligned}\epsilon_x =& e^{-3t}[(X \cos t + Y \sin t],\\ \epsilon_y =& \frac{1}{2}e^{-3t}[-(X + Y)\cos t + (X - Y)\sin t],\end{aligned}$$

where the small deviations from equilibrium ϵ_x, ϵ_y are defined by $x = 1 + \epsilon_x$, $y = 1 + \epsilon_y$, and where X, Y are constants.

[20 marks]